

# ALGEBRAIC TOPOLOGY

SYLVAIN GAULHIAC AND BINGYU ZHANG

References and textbooks: [Vic94; Hat02; Kam22]. Further reading: [May99; Die08; GJ09; BT82; Bre93]. For the required homological algebra preliminary, we refer to [Wei94, Chapter 1-3], and a cheat sheet is in Appendix A.

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## PREFACE

These are the lecture notes for Algebraic Topology, taught in the summer term of 2026 at the Kyiv School of Economics for master students. In this lecture, we try to give an introduction to basic notions and applications for algebraic topology. Due to time constraints, we will not try to cover too many details, but at least we will state many results precisely.

However, we also do not want students to restrict themselves on very standard topics. So, we try to present materials in a way that is easier to relate to some further aspects of algebraic topology that we didn't carefully discuss in the lecture.

Here are some examples: 1) We try to emphasize more about the simplicial set structure of singular simplexes, which gives a first glimpse of simplicial homotopy theory and  $\infty$ -category in an introductory course for algebraic topology; 2) We focus on the Mayer-Vietoris and mapping cone approach for the long exact sequence rather than the relative homology approach. The reason is that those tools are somewhat closer to sheaf theory and stable  $\infty$ -categories (the  $\infty$ -version of triangulated categories). In the process, we may see more about how to use the notion of cofibration without going too much into actual homotopy theory. Especially, we will construct the cellular homology without relative homology, which essentially has no big difference but sees a little more on the space level.

In addition, we make such an attempt by state supplement comments on some content about their related notion in different/further directions (which means that you can safely skip them). We will see some short discussions on de Rham theory, some ideas on  $\mathbb{E}_\infty$ -algebra when discussing the cup product, and many others.

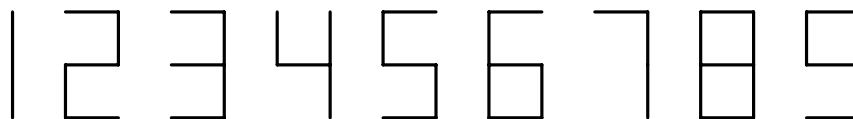
On the other hand, we put many technical discussions in exercises, but we try to organize all those materials easy to follow by students themselves by subdividing the big results into more step-by-step questions. We hope students will not lose too much technical training by carefully completing all those exercises.

Theoretically, our students have basic knowledge about homological algebra, so the discussion didn't show too much in the main content, but we still prepared a cheat sheet for the convenience of readers.

## PHILOSOPHICAL INTRODUCTION

Algebraic topology aims to use **invariants** constructed from abstract algebra to help us detect **different** topological spaces. Let's start with one example.

**Example.** Consider the digits with the following font



We treat them as subspaces  $A$  of  $\mathbb{R}^2$  (i.e., the blackboard). Could you classify them up to homeomorphisms?

Answer: They form 4 homeomorphic classes (for simplicity, we use the usual font to represent them)

$$\{1, 2, 5, 7\}, \{6, 9\}, \{3, 4\}, \{8\}.$$

For the class  $\{1, 2, 5, 7\}$ , it is direct to construct homeomorphisms between them, and all of them are homeomorphic to the interval  $[0, 1]$ .

For the class  $\{6, 9\}$ , they are related by a rotation by  $\pi$ .

For the class  $\{3, 4\}$ , they are homeomorphic to a letter  $Y$ .

However, how to show they are different classes? First tool, we consider we can consider the number of (path) connected components after removing one point from  $A$ . Then you can distinguish the class  $\{1, 2, 5, 7\}$  and  $\{3, 4\}$ . Second tool, you can consider how many loops are in  $A$ .

You may abstract the strategy into graph theory. But we are thinking differently, we are more interested in what kind of invariant we can extract from topology! For example, here we use

$$|\pi_0(A \setminus a)| = |\{\text{Path components of } A \setminus a\}|,$$

and

$$b_1(A) = \text{“The number of (independent) loops in } A\text{.”}$$

Both of them can be constructed from the invariant we will develop in the course!

The next question: for the two classes  $\{1, 2, 5, 7\}$  and  $\{3, 4\}$ , if we allow them to be manipulated like play-dough (for example, we can flatten and stretch), then these two classes cannot be distinguished.

It tells us one important idea: It is crucial to tell the precise meaning of **different!** It means you need to have a standard to distinguish spaces. Moreover, sometimes, it makes things easier if you allow a more flexible classification standard!

- Whether something is an invariant or not is sensitive to your standard.
- In the future, you will find that the idea you can learn from the course can be applied not only to topological spaces, but also to algebraic varieties, or many other things (for example, data, where there exists a subject called topological data analysis).

In this course, we will mainly focus on

### Homotopy invariants such as homology, cohomology, and homotopy groups of spaces.

With them, we will see some of the very famous theorems are proven easily using homology theory:

- (1) Brouwer fixed point theorem: A continuous map between  $D^n$  must have a fixed point.
- (2) Dimension invariance: Let  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^m$  are open sets. Suppose  $U$  and  $V$  are homeomorphisms, then  $m = n$ .

*Remark.* It is possible to define certain homeomorphic invariants, or diffeomorphic invariants. However, it would be hard to define and compute them. It is a compromise that we focus on homotopy invariants. They are rough in some sense, however, computable (but still hard in general). But anyway: Good invariants are those both computable and distinguish enough things! It means that you have to give up something to make life easier and more effective.

## 1. HOMOTOPY AND HOMOTOPIC EQUIVALENCE

Now, we go to some actual math.

In this course, we always equip  $I = [0, 1]$  with the usual topology. We set  $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$  and  $D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| \leq 1\}$ , where  $|x|$  is the 2-norm of  $x$ .

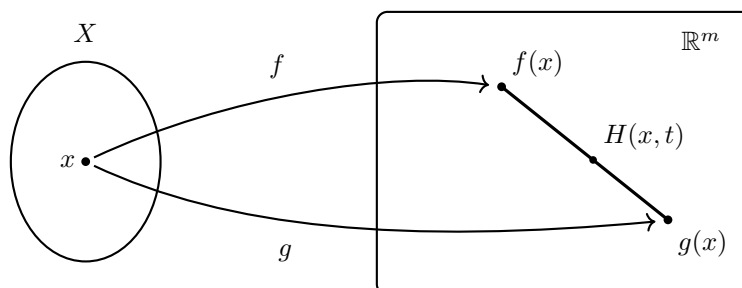
All spaces are assumed to be topological spaces, and all maps are assumed to be continuous (in case I forget to mention topological/continuity).

**Definition 1.1.** Let  $f, g : X \rightarrow Y$  be two continuous maps. We say they are homotopic if there exists a continuous map  $H : X \times I \rightarrow Y$  such that  $H(x, 0) = f(x)$  and  $H(x, 1) = g(x)$ , and we say the map  $H$  is a homotopy between  $f, g$ . We often denote  $f \stackrel{H}{\simeq} g$ , or  $f \simeq g$  if  $H$  is clear.

**Exercise 1.2.** Show that the homotopic relation between maps defines an equivalence relation on the set of continuous maps  $C^0(X, Y)$ . We often write  $[X, Y] = C^0(X, Y)/hntp$ , and a homotopy class is denoted by  $[f]$ . For composable maps, we may define  $[g] \circ [f] := [g \circ f]$ , and show that this is well-defined.

**Example 1.3.** Let  $f, g : X \rightarrow \mathbb{R}^m$  be two continuous maps, then they are homotopic. In fact, we can set the straight line homotopy

$$H(x, t) = (1 - t)f(x) + tg(x).$$



Notice that it is important that  $\mathbb{R}^m$  is linear, otherwise it does not make sense for the formula.

**Example 1.4.** Set  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ . Let  $f, g : S^1 \rightarrow S^1$  where  $f$  is the identity map, and  $g$  is the constant map  $g(z) = 1$ . Then we have  $f$  and  $g$  are not homotopic. We are not able to prove it at this time.

But you can learn from the example that the homotopic relation is sensitive to the codomain of the maps since  $f, g$  are indeed homotopic if we think of them as maps into  $\mathbb{R}^2$ .

**Definition 1.5.** We say two topological spaces are homotopy equivalent if there exists  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that  $fg \simeq \text{id}_Y$  and  $gf \simeq \text{id}_X$ .

**Exercise 1.6.** Show that the homotopy equivalence relation between spaces defines an equivalence relation on the set of all topological spaces.

If the two spaces are homotopy equivalent, then we say they have the same homotopy type, and in the same manner, we say one equivalent class of spaces with respect to the homotopy equivalence relation is a homotopy type.

*Remark 1.7.* It is clear that if two spaces are homeomorphic, then they are homotopy equivalent (i.e., have the same homotopy type). But the converse is not true, for example,  $\mathbb{R}^m$  ( $m \geq 1$ ) is homotopy equivalent to a point by Example 1.9 below, but  $\mathbb{R}^m$  is not homeomorphic to a point since they have different cardinality.

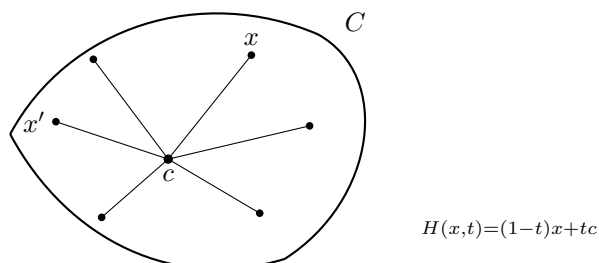
You may think that the homotopy equivalence relation is too flexible. However, you will see in the future that classifying spaces by homotopy equivalence is an impossible mission (but still more approachable than homeomorphic classification)

**Definition 1.8.** A non-empty space  $X$  is said to be contractible if it is homotopy equivalent to a point, equivalently, if it has the homotopy type of a point.

**Example 1.9.** Let  $C \subset V$  be a convex set in a real topological vector space  $V$  (for example  $V = \mathbb{R}^n$ ). We claim that for any  $c \in C$ , the two spaces  $C$  and  $\{c\}$  are homotopy equivalent. In particular, convex sets are contractible.

In fact, let  $i : \{c\} \rightarrow C$  be the inclusion, and we define  $r : C \rightarrow \{c\}$  as the constant map. Then we have  $ri = \text{id}_{\{c\}}$ , and  $f = ir : C \rightarrow C$  is given by  $f(x) = c$  (but whose codomain is  $C$  not  $\{c\}$ ). We conclude by showing that  $f \simeq \text{id}_C$ .

We can actually use the linear homotopy constructed in Example 1.3, which works since  $C$  is a convex set.



**Exercise 1.10.** Show 1) a space  $X$  is contractible if and only if the  $\text{id}_X$  is homotopic to a constant map  $c_{x_0} : X \rightarrow X$  for  $x_0 \in X$ ; 2) a contractible space is path-connected; 3) Similar to Example 1.3, show that if  $Y$  is contractible, then any two continuous map  $f, g : X \rightarrow Y$  are homotopic.

*Remark 1.11.* Variants of the definitions

(1) For  $f, g : X \rightarrow Y$ , and a subspace  $A \subset X$ , we say  $f \simeq_A g$ , and say  $f$  is homotopic to  $g$  relative to  $A$ , if  $f \simeq g$  and  $H(a, t) = f(a) = g(a)$  for  $(a, t) \in A \times I$ .

(2) For a topological space  $X$  and a subspace  $A \subset X$ , we consider the space pair  $(X, A)$ . A map  $f : (X, A) \rightarrow (Y, B)$  means a map  $f : X \rightarrow Y$  such that  $f(A) \subset B$ .

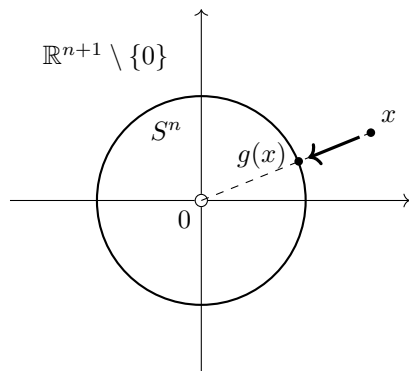
Then one can define homotopic maps and homotopy equivalence between pairs similarly.

It is often (but NOT always) the case that during a topology space is homotopy equivalent to a subspace. We have the definition:

**Definition 1.12.** A subspace  $i : A \subset X$  is a deformation retract of  $X$  if there is a retraction  $g : X \rightarrow A$  (i.e., a map  $g : X \rightarrow A$  with  $g(a) = a$  for every  $a \in A$ ) such that  $ig \simeq \text{id}_X$ .

A subspace  $A$  of  $X$  is a strong deformation retract if there is a retraction  $g : X \rightarrow A$  such that  $ig \stackrel{H}{\simeq} \text{id}_X$  through a homotopy  $H$  with the property  $H(a, t) = a$  for all  $(a, t) \in A \times I$ . Precisely, i.e.  $ig \stackrel{H}{\simeq} \text{id}_X$  relative to  $A$ .

**Example 1.13.** Let  $X = \mathbb{R}^{n+1} \setminus \{0\}$ , and  $A = S^n$ . Then we have  $g(x) = x/|x|$  gives a retraction that makes  $A$  a strong deformation retract of  $X$ , where we may take  $H(x, t) = |x|^{t-1}x : X \times I \rightarrow X$ .

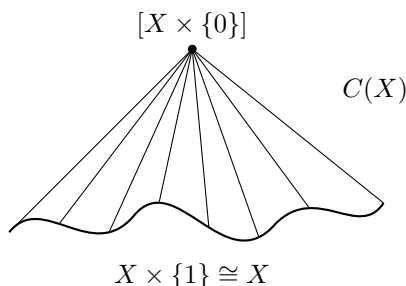


**Exercise 1.14.** Similar to the argument of Example 1.9, show that if  $A$  is a deformation retract of  $X$ , then  $A$  is homotopy equivalent to  $X$ .

**Exercise 1.15.** Let  $X$  be a topological space, and we define the cone of  $X$  as the quotient space  $C(X) := X \times [0, 1] / X \times \{0\}$ .

Show that 1)  $C(S^n) \cong D^{n+1}$  (homeomorphic) where  $S^n$  and  $D^{n+1}$ , 2) For any  $X$ , we have  $C(X)$  is contractible. In particular,  $D^{n+1}$  is contractible.

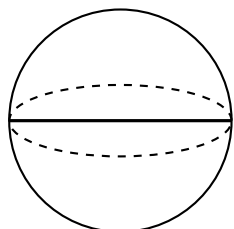
Hint: Try to show that  $[X \times \{1\}] \in C(X)$  is a strong deformation retract of  $C(X)$ . You may also convince yourself by seeing what happened for  $X = S^n$ .



**Exercise 1.16.** Try to convince yourself that the following spaces are all homotopy equivalent. If it is possible, write down homotopies between the relevant maps.

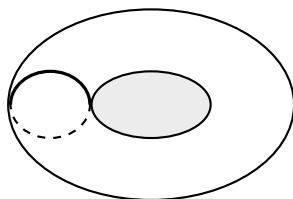
- A  $S^2$  adding a diameter.
- A torus  $T^2 = S^1 \times S^1$  glued with a disk along the meridian.
- A  $S^2$  glued with a tangential  $S^1$ .

$S^2$  with a diameter



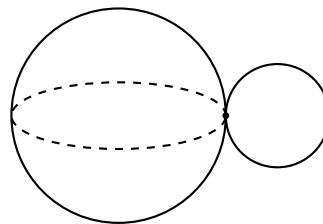
(1)

$T^2$  with a disk glued along a meridian



(2)

$S^2$  with a tangential  $S^1$



(3)

**Definition 1.17.** Let  $(X_i, x_i)_{i \in I}$  be a  $I$  family of pointed spaces, i.e.  $x_i \in X_i$  is a given point for every  $i \in I$ . We define

$$\bigvee_{i \in I} X_i = \bigsqcup_{i \in I} X_i / \sim$$

where the relation is asking all base points  $(x_i, i) \in \bigsqcup_{i \in I} X_i$  are identified to a single new marked point  $\text{pt}$  and equipping the quotient topology on it. Then the pointed space  $(\bigvee_{i \in I} X_i, \text{pt})$  is called the wedge sum (or one-point union, or simply wedge) of the family  $(X_i, x_i)_{i \in I}$ .

For example, all of 3 spaces in the previous example are homotopy equivalent to  $S^2 \vee S^1$  for some marked points. Or  $S^n / \sim$  by collapsing the equator is homeomorphic to  $S^n \vee S^n$ .

**Exercise 1.18.** We consider the infinite-dimensional sphere: Consider the space of square summable sequences  $\ell^2(\mathbb{C})$ , and set

$$S^\infty := \{(c_i)_{i \in \mathbb{N}} \mid c_i \in \mathbb{C}, \sum_{i \in \mathbb{N}} |c_i|^2 = 1\}.$$

Show that  $S^\infty$  is contractible.

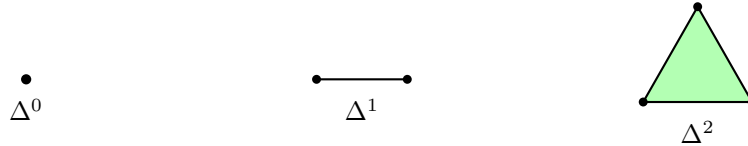
Notice that: we will see later that for each  $n \geq 1$ ,  $S^n$  is not contractible.

2. SINGULAR HOMOLOGY

2.1. **Singular simplexes.** We start from the combinatorial structure of topological simplexes.

**Definition 2.1.** Let  $q \geq 0$  be an integer, we set  $q$ -simplex as the topological space (with the subspace topology).

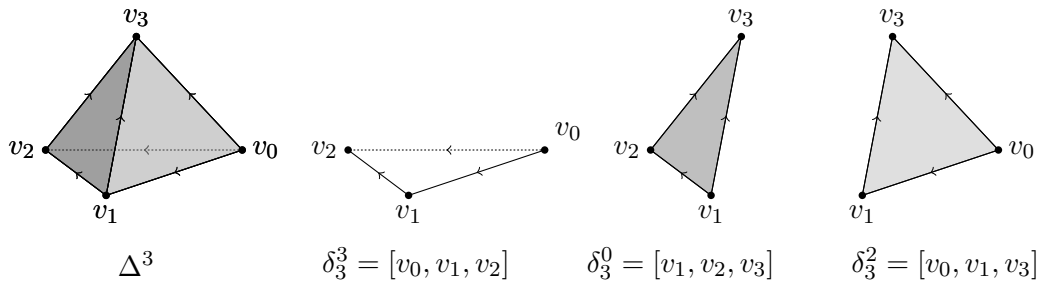
$$\Delta^q = \{(t_0, \dots, t_q) \in \mathbb{R}^{q+1} \mid \forall i, t_i \geq 0, \sum_{i=0}^q t_i = 1\}.$$



**Example 2.2.** For example, we have  $\Delta^1 \cong I = [0, 1]$ . Precisely,  $\Delta^1 = \{(1-t, t) \mid t \in [0, 1]\} \cong \{t \mid t \in [0, 1]\}$ .

It is easy to see that  $\Delta^q \cong D^q$  for all  $q$ . All  $\Delta_q$  are convex, so they are contractible.

We set  $v_i$  for  $i = 0, \dots, q$  to be the vertices of  $\Delta^q$  where only  $t_i = 1$  and all others are zero. With the notation, we often write a simplex by  $[v_0, \dots, v_q]$ .



It is important that  $\Delta^q$  carries many combinatorial information: Fix  $q \geq 1$ , for all  $i = 0, \dots, q$ , we can define a continuous map (be careful when  $i = 0$ )

$$\delta_q^i : \Delta^{q-1} \rightarrow \Delta^q, \quad \delta_q^i(t_0, \dots, t_{q-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{q-1}).$$

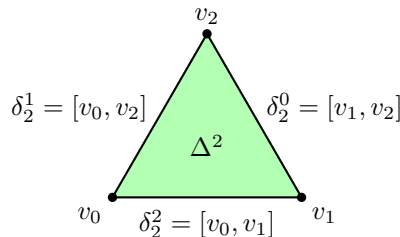
We call  $\delta_q^i$  or its image the  $i$ -th face of  $\Delta^q$ . If you draw the picture, the name motives very well!

If  $q$  is clear, we often write  $\delta^i = \delta_q^i$ . Using the notation  $[v_0, \dots, v_q]$ , we can write  $\delta_q^i = [v_0, \dots, \hat{v}_i, \dots, v_q]$ .

**Example 2.3.** When  $q = 1$ , we have  $\delta_1^0 = \{(0, 1)\} \subset \Delta^1$ , and  $\delta_1^1 = \{(1, 0)\} \subset \Delta^1$ .

Under the identification  $\Delta^1 = \{(1-t, t) \mid t \in [0, 1]\} \cong \{t \mid t \in [0, 1]\}$ . We have  $\delta_1^0$  is identified with  $1 \in [0, 1]$  and  $\delta_1^1$  is identified with  $0 \in [0, 1]$ .

When  $q = 2$ , you can see the picture below, and  $q = 3$  above (space reason there is no  $\delta_3^3$ )



On the other hand, we shall consider, for  $i = 0, \dots, q$ ,

$$\sigma_q^i : \Delta^{q+1} \rightarrow \Delta^q, \quad \sigma_q^i(t_0, \dots, t_q) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_q).$$

They are called elementary degenerate simplexes: Degeneration in the sense that the domain has a higher dimension than the range, and some parts of  $\Delta^q$  are collapsed in a certain way. Similar to faces, we often write  $\sigma^i = \sigma_q^i$ .

**Exercise 2.4.** (1) For every  $q \geq 2$  and every  $0 \leq i < j \leq q$ ,

$$\delta_q^j \circ \delta_{q-1}^i = \delta_q^i \circ \delta_{q-1}^{j-1} \quad : \Delta^{q-2} \rightarrow \Delta^q.$$

(2) For every  $q \geq 0$  and every  $0 \leq i \leq j \leq q$ ,

$$\sigma_q^j \circ \sigma_{q+1}^i = \sigma_q^i \circ \sigma_{q+1}^{j+1} \quad : \Delta^{q+2} \rightarrow \Delta^q.$$

(3) For every  $q \geq 1$ , every  $0 \leq i \leq q$ , and every  $0 \leq j \leq q+1$ ,

$$\sigma_q^j \circ \delta_{q+1}^i = \begin{cases} \delta_q^i \circ \sigma_{q-1}^{j-1}, & i < j, \\ \text{id}_{\Delta^q}, & i = j \text{ or } i = j + 1, \\ \delta_q^{i-1} \circ \sigma_{q-1}^j, & i > j + 1, \end{cases} \quad : \Delta^q \rightarrow \Delta^q.$$

A magic is that those data helps us to encode all homotopic information for a given topological space.

**Definition 2.5.** For a given topological space  $X$ , a **singular  $q$ -simplex** is a continuous map:  $\Delta^q \rightarrow X$ . We denote the set of singular  $q$ -simplexes as

$$\text{Sing}(X)_q := C^0(\Delta^q, X).$$

For a singular simplex  $f : \Delta^q \rightarrow X$ , we also often write  $f = [f_0, \dots, f_q]$  in a similar (but with a different meaning from previous notations), where  $f_q = f(v_q)$ .

By pre-composing with faces and degenerations (for suitable  $q, i$ ), we define

$$d_q^i : \text{Sing}(X)_q \rightarrow \text{Sing}(X)_{q-1}, \quad f = [\Delta^q \rightarrow X] \mapsto d_q^i(f) = f \circ \delta_q^i = [\Delta^{q-1} \xrightarrow{\delta_q^i} \Delta^q \xrightarrow{f} X],$$

$$s_q^i : \text{Sing}(X)_q \rightarrow \text{Sing}(X)_{q+1}, \quad f = [\Delta^q \rightarrow X] \mapsto s_q^i(f) = f \circ \sigma_q^i = [\Delta^{q+1} \xrightarrow{\sigma_q^i} \Delta^q \xrightarrow{f} X].$$

With the bracket notation, we have  $d_q^i(f) = [f_0, \dots, \hat{f}_i, \dots, f_q]$  for simplicity.

As a consequence of Exercise 2.4, we have

(1) For every  $q \geq 2$  and every  $0 \leq i < j \leq q$ ,

$$d_q^j \circ d_{q-1}^i = d_q^i \circ d_{q-1}^{j-1}.$$

(2) For every  $q \geq 0$  and every  $0 \leq i \leq j \leq q$ ,

$$s_q^j \circ s_{q+1}^i = s_q^i \circ s_{q+1}^{j+1}.$$

(3) For every  $q \geq 1$ , every  $0 \leq i \leq q$ , and every  $0 \leq j \leq q+1$ ,

$$s_q^j \circ d_{q+1}^i = \begin{cases} d_q^i \circ s_{q-1}^{j-1}, & i < j, \\ \text{id}_{\Delta^q}, & i = j \text{ or } i = j + 1, \\ d_q^{i-1} \circ s_{q-1}^j, & i > j + 1. \end{cases}$$

**Warning:** In this course, we will only use faces maps  $d_i$  as well as the equations

$$d_q^j \circ d_{q-1}^i = d_q^i \circ d_{q-1}^{j-1}.$$

So feel free to ignore degeneration during this course. But the role of degeneration will somehow be explained in the following.

*Supplement material 2.6.* Those commuting relations means that all those data  $\text{Sing}(X)_\bullet = (\text{Sing}(X)_q, d_q^i, s_q^j)_{q,i,j}$  form a simplicial set. (And all faces data  $(\text{Sing}(X)_q, d_q^i)_{q,i}$  form a semi-simplicial set.)

Moreover, one can show that  $\text{Sing}(X)_\bullet$  is a so-called Kan complex (in particular a quasi-category), which is a model for  $\infty$ -groupoid (in particular  $\infty$ -category).

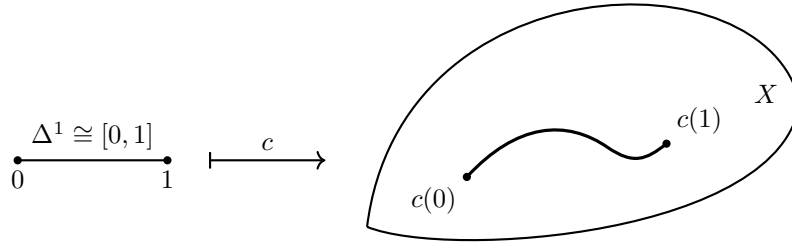
It is proven by Milnor that you can learn all the information of  $X$  as a homotopy type from  $\text{Sing}(X)_\bullet$ .

Let us see some examples.

**Example 2.7.** For any curve  $c : [0, 1] \rightarrow X$ , we may naturally define a singular 1-simplex

$$c : \Delta^1 \cong [0, 1] \xrightarrow{c} X.$$

As explained in Example 2.3, we have  $d^0(c) = c(1) \in X$  and  $d^1(c) = c(0) \in X$ .

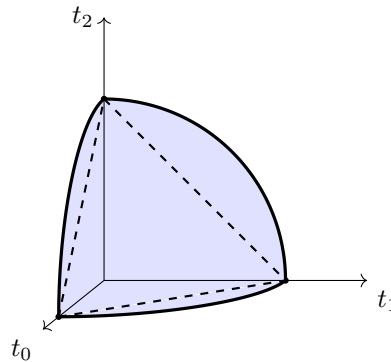


**Example 2.8.** Consider  $X = S^n$ , we define

$$f : \Delta^n \rightarrow S^n, \quad t \in \Delta^n \mapsto t/|t|.$$

Similarly, for  $I \subset \{0, \dots, n\}$ , we set  $t_I = ((-1)^{\chi_I(i)} t_i)$  ( $\chi_I$  is the indicator function of the set  $I$ ), and  $\sigma_I(t) = t_I/|t_I| \in S^n$ . Then we obtain  $2^{n+1}$  many  $n$ -simplexes in  $S^n$ .

You may also easily describe their boundary.



*Remark 2.9.* Historically, the adjective “singular” means that, compared to literally topological simplexes, you can allow really weird simplexes.

For example, Constant maps  $\Delta^q \rightarrow X$  are singular simplexes. The Peano curve  $\Delta^1 = I \rightarrow \mathbb{R}^2$  is also a singular simplex.

**2.2. Singular chain and singular homology.** Seems that what we do just repeats certain geometric constructions and no algebra yet. Here it is!

**Recall:** For any set  $S$ , there exists an abelian group freely generated by  $S$ , say

$$\mathbb{Z}[S] = \bigoplus_{x \in S} \mathbb{Z}x = \{(n_x)_{x \in S} \mid \text{finitely many non-zero } n_x\}.$$

If  $m : S \rightarrow T$  a map, we can define  $\mathbb{Z}[m] : \mathbb{Z}[S] \rightarrow \mathbb{Z}[T]$  by assign  $\mathbb{Z}[m](x) = m(x)$  and extend linearly  $\mathbb{Z}[m](\sum_x n_x x) = \sum_x n_x m(x)$ . Then  $\mathbb{Z}[m]$  is a group homomorphism between abelian groups.

Here, we shall apply the construction to  $\text{Sing}(X)_q$  and  $d_q^i$ :

**Definition 2.10.** We define the abelian group of singular chains as

$$S_q(X) := \mathbb{Z}[\text{Sing}(X)_q]$$

and  $\mathbb{Z}$ -linear faces maps as group homomorphisms

$$\partial_q^i := \mathbb{Z}[d_q^i] : S_q(X) \rightarrow S_{q-1}(X).$$

By definition, a singular  $q$ -chain is a formal linear combination of finitely many singular  $q$ -simplexes

$$\sigma = n_1\sigma_1 + \cdots + n_k\sigma_k, \quad n_i \in \mathbb{Z}, \sigma_i \in \text{Sing}(X)_q.$$

*Remark 2.11.* It is explained in Definition 2.5, we may consider  $\mathbb{Z}[s_q^i]$  as well. In this case, we get a simplicial abelian group  $(S_q(X), \partial_q^i, \mathbb{Z}[s_q^i])$ , as well as a corresponding semi-simplicial abelian group  $(S_q(X), \partial_q^i)$ .

**Lemma 2.12.** *We define*

$$\partial_q = \sum_i (-1)^i \partial_q^i : S_q(X) \rightarrow S_{q-1}(X),$$

*then it satisfies that  $\partial_{q-1}\partial_q = 0$ . In particular, we set  $S_q(X) = \{0\}$  for  $q < 0$ , and set  $\partial_q = 0$  for  $q \leq 0$ ; then  $S_*(X) = (S_q(X), \partial_q)$  form a chain complex (in homological degree convention).*

*Proof.* You will see that the result is a purely algebraic corollary of the faces relation  $d_q^j \circ d_{q-1}^i = d_q^i \circ d_{q-1}^{j-1}$ .

We only need to verify the equation  $\partial_{q-1}\partial_q = 0$  on generators of  $S_n(X)$  (i.e., singular simplexes). For a singular simplex  $\sigma : \Delta^q \rightarrow X$ , recall the definition  $\partial_q^i(\sigma) = d_i^q(\sigma)$ .

Hence

$$\partial_{q-1}\partial_q(\sigma) = \sum_{i=0}^q \sum_{j=0}^{q-1} (-1)^{i+j} d_i^q \circ d_j^{q-1}(\sigma).$$

Using the faces relation  $d_q^j \circ d_{q-1}^i = d_q^i \circ d_{q-1}^{j-1}$ . We see that each term with indices  $(i, j)$  such that  $j < i$  is equal to the term with indices  $(j, i-1)$ , but with the opposite sign  $(-1)^{i+j} = -(-1)^{j+i-1}$ .

Therefore, all terms cancel in pairs, and so

$$\partial_{q-1}\partial_q(\sigma) = 0. \quad \square$$

*Supplement material 2.13.* The proof Lemma 2.12 only uses the face relation, and nothing is about the topology except the constructions. This is a pattern of a general theorem that relates simplicial abelian groups and chain complexes: *Dold-Kan correspondence*. We have an equivalence between the category of simplicial abelian groups and the category of (non-negatively homological graded) chain complexes,

$$\text{sAb} \cong \text{Ch}_{\geq 0}(\mathbb{Z}).$$

**Definition 2.14.** For a topological space  $X$ , we define the **singular chain** complex as the following chain complex  $S_*(X) = (S_q(X), \partial_q)$ :

$$\cdots \rightarrow S_{q+1}(X) \xrightarrow{\partial_{q+1}} S_q(X) \xrightarrow{\partial_q} S_{q-1}(X) \rightarrow \cdots.$$

We set abelian groups

$$B_q(X) := \text{im}(\partial_{q+1}) \subset \ker(\partial_q) =: Z_q(X)$$

and define the  $q$ -th **singular homology** group as the quotient

$$H_q(X) := Z_q(X)/B_q(X).$$

We call chains in  $Z_q(X)$  cycles and chains in  $B_q(X)$  boundaries. Elements in  $H_q(X)$  as homology classes. If two cycles  $\sigma_1, \sigma_2$  are in the same homology class, i.e.  $\sigma_1 - \sigma_2 = \partial_{q+1}\tau$ , we say they are homologous.

As usual, we will simply write  $\partial = \partial_q$  if  $q$  is clear.

Notice that we do not emphasize  $q \geq 0$  since we get trivial data when  $q < 0$ .

*Remark 2.15.* Compare to  $\text{Sing}(X)_q$ , the key point for  $S_q(X)$  is that it is an abelian group, so we can actually do “linear algebra” to perform computation! In fancy words,  $S_q(X)$  are linearization of  $\text{Sing}(X)_q$ .

However,  $S_q(X)$  is a huge abelian group. It is impossible to compute the singular homology by definition. We need to develop sufficient computational tools later, such as the Mayer-Vietoris sequence and the excision principle.

The advantage of singular homology is that it is easier to construct the entire homology theory.

On the other hand, we will see that  $S_*(X)$  lose some information compare to  $\text{Sing}(X)_\bullet$  known all the homotopic information of  $X$ .

**Example 2.16.** A continuation of Example 2.7. For a curve  $c : I \rightarrow X$ , we regard it as a singular 1-simplex. Then we can also regard it as a singular 1-chain  $c \in S_1(X)$ . Then we have

$$\partial c = \partial^0(c) - \partial^1(c) = d^0(c) - d^1(c) = c(1) - c(0)$$

where we regard points  $c(i)$  as singular 0-simplexes.

Consequently,  $c$  is a cycle if and only if  $c$  is a closed curve, i.e.,  $c(1) = c(0)$ .

**Exercise 2.17.** Let  $C \subset \mathbb{R}^n$  be a convex set. Take  $x, y \in C$ . We denote  $[x, y]$  (resp.  $[y, x]$ ) the linear singular simplex defined by the segment  $I \rightarrow C$  from  $x$  to  $y$  (resp. from  $y$  to  $x$ ).

Show that  $[x, y] \neq -[y, x]$  in  $S_1(C)$ , but  $[x, y] + [y, x]$  is homologous to the zero cycle 0, i.e.  $[x, y] + [y, x] = \partial_2 \sigma$  for some  $\sigma \in S_2(X)$ .

**Exercise 2.18.** A continuation of Example 2.8. Let  $X = S^n$ , in this example, we construct a singular  $n$ -simplex  $\sigma_I(t) = t_I/|t_I| \in S^n$  where  $I \subset \{0, \dots, n\}$  and  $t_I = ((-1)^{\chi_I(i)} t_i)$  ( $\chi_I$  is the characteristic function of the set  $I$ ).

Then we have

$$\sigma_{S^n} := \sum_I (-1)^{|I|} \sigma_I.$$

Show that  $\sigma_{S^n}$  is a cycle. Hint: you can start by understanding  $n = 1, 2$  cases.

Later, we will see that  $[\sigma_{S^n}]$  is called a fundamental class of  $S^n$ , which is a generator of  $H_n(S^n) \cong \mathbb{Z}$ !

### 2.3. First computation and properties.

**Proposition 2.19.** *Singular homology for a point  $X = \text{pt}$ . We have*

$$H_0(\text{pt}) \cong \mathbb{Z}, \quad H_q(\text{pt}) = 0, \forall q \geq 1.$$

*Proof.* The idea is that the only possible singular  $q$ -simplex is the constant map  $c_q$ , so we have  $\text{Sing}(\text{pt})_q = \{c_q\}$  is a single point set, then we have  $S_q(\text{pt}) = \mathbb{Z}c_q$  for  $q \geq 0$ . So, the singular chain complex is of the form

$$\dots \rightarrow \mathbb{Z}c_{q+1} \xrightarrow{\partial_{q+1}} \mathbb{Z}c_q \xrightarrow{\partial_q} \mathbb{Z}c_{q-1} \rightarrow \dots \xrightarrow{\partial_1} \mathbb{Z}c_0 \rightarrow 0 \rightarrow 0 \rightarrow \dots.$$

It remains to compute  $\partial_q$ . Since  $\text{Sing}(\text{pt})_{q-1}$  also only consists of constant map, we have that  $d_q^i(c_q) : \Delta^{q-1} \rightarrow \text{pt}$  are all the same map  $c_{q-1}$ . Consequently, we have

$$\partial_q c_q = \sum_i (-1)^i \partial_q^i(c_q) = \sum_i (-1)^i c_{q-1} = c_{q-1} - c_{q-1} + c_{q-1} - c_{q-1} + \dots.$$

Therefore, we have

$$\partial_{2i-1} c_q = 0, \quad \partial_{2i} c_q = c_{q-1},$$

and the singular complex becomes

$$\dots \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \rightarrow \dots \xrightarrow{0} \mathbb{Z} \rightarrow 0 \rightarrow 0 \rightarrow \dots.$$

Consequently, we have

$$H_0(\text{pt}) \cong \mathbb{Z}, \quad H_q(\text{pt}) = 0, \forall q \geq 1. \quad \square$$

**Proposition 2.20.** *For any topological space  $X$ , we have for all  $q$*

$$H_q(X) \cong \bigoplus_{X_\alpha \in \pi_0(X)} H_q(X_\alpha).$$

*Proof.* In fact, since  $\Delta_q$  is path connected. Then we have

$$\text{Sing}(X)_q = \bigsqcup_{X_\alpha \in \pi_0(X)} \text{Sing}(X_\alpha)_q.$$

And the same reason shows that  $d^i$  respects the components decomposition. Consequently, we have the decomposition of chain complex

$$S_*(X) \cong \bigoplus_{X_\alpha \in \pi_0(X)} S_*(X_\alpha),$$

which induces decomposition on homology.  $\square$

**Exercise 2.21.** 1) Complete the detail of Proposition 2.20. 2) Based on the same idea, show that for all  $q$

$$H_q\left(\bigsqcup_i X_i\right) \cong \bigoplus_i H_q(X_i).$$

**Proposition 2.22.** *If  $X$  is path connected, then we have  $H_0(X) = \mathbb{Z}x$ , where  $x$  means either an (arbitrary) base point or the singular 0-simplex give by  $x$ .*

*Consequently, for all spaces  $X$ , we have*

$$H_0(X) \cong \bigoplus_{X_\alpha \in \pi_0(X)} \mathbb{Z}.$$

*Proof.* Consider the augmentation homomorphism

$$\epsilon : S_0(X) \rightarrow \mathbb{Z}, \quad \sum_i n_i p_i \mapsto \sum_i n_i,$$

where points  $p_i$  are regarded as 0-simplexes.

It is clear that  $\epsilon$  is surjective for all spaces. However, we shall prove that if  $X$  is path-connected, then

$$\ker(\epsilon) = \text{im}(\partial_1).$$

Proof of the equality: it is clear that  $\text{im}(\partial_1) \subset \ker(\epsilon)$ . Conversely, take  $a \in \ker(\epsilon)$  and we write  $a = \sum_i n_i p_i$ , then we have  $0 = \epsilon(a) = \sum_i n_i$ . For a fixed point  $x \in X$ , we pick paths  $\gamma_i$  from  $x$  to  $p_i$ , which is possible since  $X$  is path-connected. Then we regard  $\gamma_i$  as 1-simplexes, and then set

$$c = \sum_i (n_i) \gamma_i.$$

You can compute that (recall Example 2.16)

$$\partial_1 c = \sum_i n_i \partial_1 \gamma_i = \sum_i n_i (p_i - x) = \sum_i n_i p_i - \left(\sum_i n_i\right) x = a \in \text{im}(\partial_1).$$

Consequently, we have the following exact sequence

$$S_1(X) \rightarrow S_0(X) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

and then  $H_0(X) \cong \mathbb{Z}x$  (try to supplement some detail).

The second statement follows from Proposition 2.20.  $\square$

**Corollary 2.23.** *A space  $X$  is path connected if and only  $H_0(X) \cong \mathbb{Z}$ .*

*Remark 2.24.* Here, by the proposition, we know that the number of path components can be computed by the rank of  $H_0$  as a free abelian group.

#### 2.4. Variant of definition.

2.4.1. *Homology with coefficient.*

**Definition 2.25.** Let  $M$  be an abelian group and  $X$  be a space, we define

$$S_*(X; M) := (S_q(X) \otimes_{\mathbb{Z}} M, \partial_q \otimes_{\mathbb{Z}} M),$$

and  $Z_q(X; M)$ ,  $B_q(X; M)$  and  $H_q(X; M)$  in the similar way. Then we call  $H_q(X; M)$  singular homology with coefficients.

*Remark 2.26.* We need the coefficient setup especially when  $M = \mathbb{F}_p$ , where some interesting chain may show up in  $\mathbb{Z}$ -coefficient. Also, certain computations would be simplified for non  $\mathbb{Z}$ -coefficient. So this is an important variant.

**Exercise 2.27.** Show 1)  $H_q(X; \mathbb{Z}) = H_q(X)$ . 2) Write down and prove all properties in Subsection 2.3 for  $H_q(X; M)$ . 3) When  $M = R$  is a commutative ring, show that  $H_q(X; R)$  are  $R$ -modules.

The following result compares homology with  $M$ -coefficient and  $\mathbb{Z}$ -coefficient.

**Theorem 2.28** (Universal coefficient theorem for homology). *Let  $M$  be an abelian group and  $X$  be a space, for each  $q$ , we have the following short exact sequence*

$$0 \rightarrow H_q(X) \otimes_{\mathbb{Z}} M \rightarrow H_q(X; M) \rightarrow \text{Tor}_1^{\mathbb{Z}}(H_{q-1}(X), M) \rightarrow 0.$$

*Proof.* Noticed that  $S_*(X)$  is a free chain complex, then you can apply the algebraic universal coefficient theorem A.17 to  $S_*(X; M) = S_*(X) \otimes_{\mathbb{Z}} M$ .  $\square$

*Remark 2.29.* If you don't know what I am talking about here, you may use it directly based on the following information: 1)  $\text{Tor}_1^{\mathbb{Z}}$  commute with arbitrary direct sum, and  $\text{Tor}_1^{\mathbb{Z}}(A, B) \simeq \text{Tor}_1^{\mathbb{Z}}(B, A)$ . 2)  $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/\text{gcd}(m, n)$ . 3)  $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, M) = 0$  if  $M$  is a flat abelian group, for example  $\mathbb{Z}$  or  $\mathbb{F}$  is a field of characteristic 0.

We mainly use the theorem for  $M = \mathbb{Z}/n$  or  $M = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ . The data shall be enough in practice.

2.4.2. *Reduced singular homology.* In the proof of Proposition 2.22, we notice that the augmentation

$$\epsilon : S_0(X; M) \rightarrow M, \quad \sum_i n_i p_i \mapsto \sum_i n_i$$

is useful.

**Definition 2.30.** Let  $M$  be a non-trivial abelian group and  $X$  be a space, we define the reduced singular chain  $\tilde{S}_*(X; M)$  as

$$\cdots \rightarrow S_{q+1}(X; M) \xrightarrow{\partial_{q+1}} S_q(X; M) \rightarrow \cdots \rightarrow S_1(X; M) \xrightarrow{\partial_1} S_0(X; M) \xrightarrow{\epsilon} M \rightarrow 0 \rightarrow \cdots,$$

i.e.  $\tilde{S}_q(X; M) = S_q(X; M)$  for  $q \geq 0$ ,  $\tilde{S}_{-1}(X; M) = M$  and  $\tilde{S}_q(X; M) = 0$  for  $q < -1$ .

Its homology groups are denoted as  $\tilde{H}_q(X; M)$ .

**Exercise 2.31.** Show that 1)  $\tilde{H}_q(X; M) = H_q(X; M)$  for  $q > 0$  and  $H_0(X; M) = \tilde{H}_0(X; M) \oplus M$  (try to write down the generator). 2)  $\tilde{H}_q(\text{pt}; M) = 0$  for all  $q \geq 0$ . 3)  $X$  is path connected if and only if  $\tilde{H}_0(X; M) = 0$ .

*Remark 2.32.* In practice, we may use the reduced homology to simplify some argument since  $\tilde{H}_0$  could very often to be trivial.

3. FUNCTORIALITY AND HOMOTOPY INVARIANCE

Previously, we defined singular homology. In this section, we study in what sense singular homologies are topological invariants.

**3.1. Pushforward maps.** Let  $f : X \rightarrow Y$  be a continuous map. Then by pre-composition, we can define

$$\text{Sing}(f)_q : \text{Sing}(X)_q \rightarrow \text{Sing}(Y)_q, \quad [\Delta^q \rightarrow X] \mapsto [\Delta^q \rightarrow X \xrightarrow{f} Y].$$

**Exercise 3.1.** Recall faces maps  $d_q^i$ . To distinguish them for different spaces, we write  $d_{q,X}^i$  and  $d_{q,Y}^i$ .

Show that

$$d_{q,Y}^i \circ \text{Sing}(f)_q = \text{Sing}(f)_{q-1} \circ d_{q,X}^i.$$

Bonus: Try to write down and prove similar statements for degeneration maps. Consequently,  $\text{Sing}(f)_\bullet : \text{Sing}(X)_\bullet \rightarrow \text{Sing}(Y)_\bullet$  is a map of simplicial sets.

**Exercise 3.2.** For two composable maps  $X \xrightarrow{f} Y \xrightarrow{g} Z$ , show that

$$\text{Sing}(g)_q \circ \text{Sing}(f)_q = \text{Sing}(g \circ f)_q.$$

Now, passing to the singular chain, we define

**Definition 3.3.** Let  $f : X \rightarrow Y$  be a continuous map. We define

$$S_q(f) := \mathbb{Z}[\text{Sing}(f)_q] : S_q(X) \rightarrow S_q(Y).$$

**Lemma 3.4.** The maps  $S_q(f)$  commute with  $\partial$ . Therefore,  $S_*(f) : S_*(X) \rightarrow S_*(Y)$  form a chain map between chain complexes.

*Proof.* Similar to Lemma 2.12, this is a consequence (linearization) of

$$d_{q,Y}^i \circ \text{Sing}(f)_q = \text{Sing}(f)_{q-1} \circ d_{q,X}^i. \quad \square$$

**Exercise 3.5.** Complete detail for Lemma 3.4.

**Definition 3.6.** With the notation, we define

$$H_q(f) : H_q(X) \rightarrow H_q(Y)$$

as  $H_q(S_*(f))$  via the formula  $H_q(f)([\sigma]) := [S_q(f)(\sigma)]$ .

(This is just the standard algebraic fact: a chain map between chain complexes induces maps on homology. Do it if you are not familiar enough with homological algebra)

*Remark 3.7. Please notice here* We will often write  $f_\# = S_*(f)$ ,  $f_{\#,q} = S_q(f)$  and  $f_q = H_q(f)$  for simplify notations.

**Proposition 3.8.** Show that 1)  $\text{Sing}(\text{id}_X)_\bullet = \text{id}_{\text{Sing}(X)_\bullet}$  and  $(\text{id}_X)_\# = \text{id}_{S_*(X)}$ .

2) For two composable maps  $X \xrightarrow{f} Y \xrightarrow{g} Z$ , show that  $g_\# \circ f_\# = (g \circ f)_\#$ , and then  $g_q \circ f_q = (g \circ f)_q$ .

3) The constant map  $f : X \rightarrow \text{pt} \rightarrow X$  induces  $H_q(f) = 0$  for  $q > 0$  and  $H_0(f) = \text{id}_{\mathbb{Z}}$  if  $X$  is path connected (try to figure out the general case).

4) Homology groups are homeomorphic invariants.

*Proof.* Term (1) is clear. The term (2) is a consequence of  $\text{Sing}(g)_q \circ \text{Sing}(f)_q = \text{Sing}(g \circ f)_q$ .

Here, we show (3). In fact, by (2), we have  $f_q$  factor through

$$f_q : H_q(X) \rightarrow H_q(\text{pt}) \rightarrow H_q(X).$$

Then by Proposition 2.19, we conclude for  $q > 0$ . For  $q = 0$ , we left as an exercise (Hint: compare to Proposition 2.22).

(4) is also an exercise. □

**Exercise 3.9.** Finish the exercise in Proposition 3.8.

*Remark 3.10.* You may know what functors are. Here, the upshot is that homology groups are functors from the category of topological spaces and continuous maps to the category of abelian groups.

**Exercise 3.11.** Write down and prove the corresponding statements for  $H_q(X; M)$ .

**3.2. Homotopy invariance.** Previously, we showed that homology groups are homeomorphic invariants. Here, we show that homology groups are actually homotopy invariants. Compared with the routine verification in the previous subsection, we require a real effect here.

First, we claim the following fact

**Theorem 3.12.** *Suppose two maps  $f, g : X \rightarrow Y$  are homotopic through  $H$ . Then there exists a chain homotopy  $h$  between  $S_*(X)$  and  $S_*(Y)$ , i.e.*

$$h_q : S_q(X) \rightarrow S_{q+1}(Y)$$

such that

$$\partial_{q+1,Y} h_q + h_{q-1} \partial_{q,X} = g_{\#,q} - f_{\#,q}.$$

$$\begin{array}{ccccccc} \longrightarrow & S_{q+1}(X) & \xrightarrow{\partial_{q+1,X}} & S_q(X) & \xrightarrow{\partial_{q,X}} & S_{q-1}(X) & \longrightarrow \\ & \searrow & & \swarrow & & \swarrow & \\ & & h_q & & f_{\#,q} & & g_{\#,q} & & h_{q-1} \\ & \swarrow & & \searrow & & \swarrow & \\ \longleftarrow & S_{q+1}(Y) & \xrightarrow{\partial_{q+1,Y}} & S_q(Y) & \xrightarrow{\partial_{q,Y}} & S_{q-1}(Y) & \longrightarrow \end{array}$$

**Corollary 3.13.** *Suppose two maps  $f, g : X \rightarrow Y$  are homotopic. Then they define the same map on the homology groups*

$$f_q = g_q : H_q(X) \rightarrow H_q(Y).$$

*In particular, if  $X$  and  $Y$  are homotopy equivalent, then  $H_q(X) \cong H_q(Y)$  for all  $q \geq 0$ .*

*Proof.* For the first statement, this is a standard homological fact: chain homotopy between chain maps induces the same map on homology groups. Do it by hand if you are not familiar with it!

For the second, we notice that by the definition of homotopy equivalence, we have  $f, g$  such that  $fg \simeq \text{id}_X$  and  $gf \simeq \text{id}_Y$ . Then we have

$$f_q g_q = (fg)_q = (\text{id}_X)_q = \text{id}_{H_q(X)}, g_q f_q = (gf)_q = (\text{id}_Y)_q = \text{id}_{H_q(Y)}. \quad \square$$

Therefore, the main technical point is the proof of Theorem 3.12.

Let me explain some ideas behind: So far, we have the homotopy

$$H : X \times I \rightarrow Y.$$

As usual, we apply Sing to  $H$  to obtain

$$\text{Sing}(X \times I)_{q+1} \xrightarrow{\text{Sing}(H)_{q+1}} \text{Sing}(Y)_{q+1}.$$

If we simply linearize it, we get a chain map  $S_{q+1}(X \times I) \rightarrow S_{q+1}(Y)$ .

However, what we want is actually  $S_q(X) \rightarrow S_{q+1}(Y)$ , which does not follow from any naive construction we have so far.

To do so, we should expect to have some maps

$$\text{Sing}(X)_q \rightarrow \text{Sing}(X \times I)_{q+1}$$

commute with faces.

It means that we should have some constructions

$$\sigma : \Delta^q \rightarrow X \quad \rightsquigarrow \quad ? : \Delta^{q+1} \rightarrow X \times I.$$

On the other hand, we have the following naive construction

$$\sigma : \Delta^q \rightarrow X \quad \rightsquigarrow \quad \sigma \times I : \Delta^q \times I \rightarrow X \times I.$$

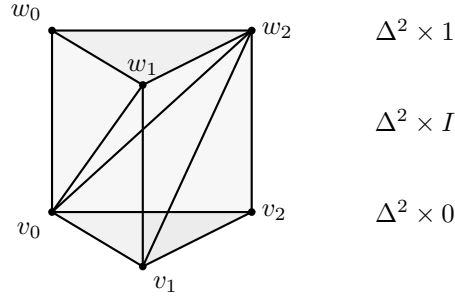
Therefore, seems that all magic is in how we divide the cylinder  $\Delta^q \times I$  into simplexes.

**Construction** [Simplicial decomposition of  $\Delta^q \times I$ ]: Here, we denote vertices in  $\Delta^q \times 0$  by  $[v_0, v_1, \dots, v_q]$  and vertices in  $\Delta^q \times 1$  by  $[w_0, w_1, \dots, w_q]$ . Then we construct a sequence of  $q + 1$ -simplexes in  $\Delta^q \times I$  for  $i = 0, \dots, q$ :

$$\tau_q^i = [v, \dots, v_i, w_i, \dots, w_q] : \Delta^{q+1} \rightarrow \Delta^q \times I.$$

Notice here, it means that we match the vertices first, and since  $\Delta^q \times I$  is convex, we can extend the maps between vertices linearly to a linear simplex.

Now, we have  $\Delta^q \times I = \cup_i \text{im}(\tau_q^i)$ . More importantly, we have certain relations. Here, let us phrase them in Sing level.



We define maps for  $i = 0, \dots, q$  by composing  $\tau_i$

$$T_q^i : \text{Sing}(X)_q \rightarrow \text{Sing}(X \times I)_{q+1}, \quad [\Delta^q \xrightarrow{\sigma} X] \mapsto [\Delta^{q+1} \xrightarrow{\tau_i} \Delta^q \times I \xrightarrow{\sigma \times I} X \times I].$$

**Exercise 3.14.** For those  $T^i$ , we have the following commutative relations.

$$d^j T^i = \begin{cases} T^{i-1} d^j, & 0 \leq j \leq i-1, \\ T^{i-1} d^i, & j = i, \\ T^i d^i, & j = i+1, \\ T^i d^{j-1}, & i+2 \leq j \leq q+1. \end{cases}$$

and

$$d^0 T^0(\sigma) = (\sigma \times \text{id}_I) \circ [w_0, \dots, w_q], \quad d^{q+1} T^q(\sigma) = (\sigma \times \text{id}_I) \circ [v_0, \dots, v_q].$$

*Supplement material 3.15.* Those data actually define a certain simplicial homotopy between simplicial maps. But we left those discussions to further reading/courses.

Now, we may linearize those maps to get

$$\mathbb{Z}[T_q^i] : S_q(X) \rightarrow S_{q+1}(X \times I)$$

and set

$$T_q = \sum_i (-1)^i \mathbb{Z}[T_q^i] : S_q(X) \rightarrow S_{q+1}(X \times I).$$

**Lemma 3.16.** For  $t \in I = [0, 1]$ , we set  $i_t : X \rightarrow X \times I, x \mapsto (x, t)$ . We have

$$\partial_{q+1, X \times I} T_q + T_{q-1} \partial_{q, X} = i_{1, \#} - i_{0, \#} : S_q(X) \rightarrow S_q(X \times I).$$

*Proof.* Similar to previous proofs, this is a corollary of Exercise 3.14. □

**Exercise 3.17.** Finish the proof of the lemma.

Now, we can prove Theorem 3.12

*Proof of Theorem 3.12.* We define

$$h_q : S_q(X) \xrightarrow{T_q} S_{q+1}(X \times I) \xrightarrow{H_{\#, q}} S_{q+1}(Y).$$

Then we have

$$\partial h + h \partial = \partial H_{\#} T + H_{\#} T \partial = H_{\#} \partial T + H_{\#} T \partial = H_{\#} (\partial T + \partial T) = H_{\#} (i_{1, \#} - i_{0, \#}) = g_{\#} - f_{\#}.$$

One small exercise for you is to think about what results we use here. □

**Exercise 3.18.** State and prove the homotopy invariance for  $H_q(X; M)$  and  $\tilde{H}_q(X; M)$ . Notice that you do not need to do all the construction again by hand. In principle, you only need to manipulate the tensor product.

**3.3. Applications.** Finally, let's see some very simple applications.

**Exercise 3.19.** Show that if  $X$  is a contractible space, then

$$H_q(X; M) = 0, \forall q \geq 1, H_0(X; M) = M.$$

In particular, now, you shall know  $H_q(D^n; M)$ ,  $H_q(C(X); M)$ ,  $H_q(C; M)$  for a convex set  $C$ .

**Exercise 3.20.** Show that if  $A \subset X$  is a deformation retraction. Then the inclusion map  $i : A \subset X$  induces group isomorphisms

$$i_q : H_q(A; M) \xrightarrow{\cong} H_q(X; M).$$

So, for example, now you shall know that  $H_q(S^n; M) \cong H_q(\mathbb{R}^{n+1} \setminus \{0\}; M)$ .

**Definition 3.21.** For a space  $X$ , if  $H_q(X)$  is a finitely generated abelian group, then the free part of  $H_q(X)$  has finite rank, we define  $b_q(X) = \text{rank} H_q(X)$ , the  $q$ -th Betti number.

We define the Euler number to be

$$\chi(X) = \sum_q (-1)^q b_q(X).$$

For example, we have  $\chi(\text{pt}) = 1$ .

**Corollary 3.22.** *Betti numbers and Euler number are homotopy invariants of spaces.*

**Exercise 3.23.** In fact, for fields  $\mathbb{K}$ , for example  $\mathbb{K} = \mathbb{F}_p, \mathbb{Q}$ , you may define the  $\mathbb{K}$ -Betti number  $\beta_q(X; \mathbb{K}) := \dim_{\mathbb{K}} H_q(X, \mathbb{K})$ .

Use the universal coefficient theorem Theorem 2.28 to show that 1) If  $\beta_q(X)$  are finite for all  $q$ ; then  $\beta_q(X; \mathbb{K})$  are finite for all  $q$ ; 2)  $\beta_q(X; \mathbb{Q}) = \beta_q(X)$  for all  $q$  when they can be defined, 3) Suppose  $\beta_q(X)$  are finite for all  $q$ . Then the Euler number  $\chi(X) = \sum_q (-1)^q b_q(X; \mathbb{K})$  for all  $\mathbb{K}$ .

*Supplement material 3.24.* However, we will see later that  $\beta_q(X; \mathbb{F}_p)$  may differ with  $\beta_q(X)$  (for example,  $X = \mathbb{R}P^2$  the real projective plane is an example). So, sometimes the  $\mathbb{F}_p$ -coefficient homology might be a more interesting invariant.

## 4. MAYER-VIETORIS SEQUENCE

In this section, we develop a major tool for computing homology: the Mayer-Vietoris sequence.

**4.1. Small chain theorem.** We start from an important technical result.

Let  $X$  be a space and  $\mathcal{U} = \{U_i \mid i \in I\}$  be a family of subspaces (NOT necessarily open, but historically we use  $U$ ) such that  $X = \cup_{i \in I} \text{Int}(U_i)$ , we define

$$\text{Sing}^{\mathcal{U}}(X)_q := \{\sigma \in \text{Sing}(X)_q \mid \exists i \in I, \text{im}(\sigma) \subset U_i\}.$$

It is clear that faces maps map  $\text{Sing}^{\mathcal{U}}(X)_q$  into  $\text{Sing}^{\mathcal{U}}(X)_{q-1}$ . So, its linearization form

$$S_*^{\mathcal{U}}(X) = (S_q^{\mathcal{U}}(X) = \mathbb{Z}[\text{Sing}^{\mathcal{U}}(X)_q], \partial_q),$$

which is a sub-chain complex of  $S_*(X)$ .

**Exercise 4.1.** We naturally think  $S_*(U_i)$  as sub-complex of  $S_*(X)$ , show that

$$S_*^{\mathcal{U}}(X) = +_{i \in I} S_*(U_i) \subset S_*(X)$$

where  $+_{i \in I}$  simply means Minkovski sum (or non-direct sum) in  $S_*(X)$ .

**Theorem 4.2** (Small chain theorem). *With the above notation, the inclusion of chain complexes*

$$i_* : S_*^{\mathcal{U}}(X) \subset S_*(X)$$

*has a chain homotopy inverse  $r_* : S_*(X) \rightarrow S_*^{\mathcal{U}}(X)$  such that  $r_* i_* = \text{id}$  and  $i_* r_*$  is chain homotopic to  $\text{id}$  through a chain homotopy  $F_*$ , and moreover  $F_*(S_*(U_i)) \subset S_{*+1}(U_i)$  for all  $i \in I$ .*

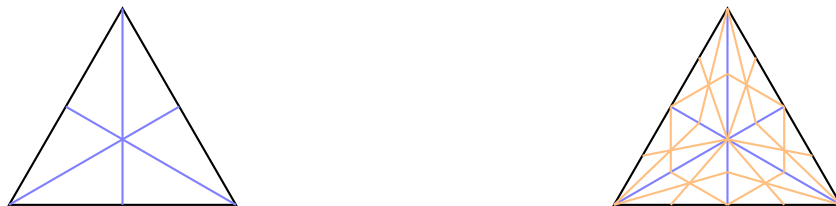
*In particular,  $i_*$  induces isomorphisms on homology groups*

$$H_q^{\mathcal{U}}(X) := H_q(S_*^{\mathcal{U}}(X)) \xrightarrow{\cong} H_q(X).$$

*Moreover,  $i_*$  commutes with the augmentation  $\varepsilon$ , and then can be extended to a chain homotopy equivalence on the reduced singular chains, and then induces isomorphisms on reduced homologies.*

*Proof.* You can find it in [Hat02, Proposition 2.21]. □

*Remark 4.3.* Its proof is very technical and long (even Hatcher uses 4 pages to prove it), and we will only try to explain the idea behind it. But the idea is very intuitive: We want to find chains small enough from an arbitrary chain but does not change homology groups. So, we may try to subdivide the  $\Delta^q$  into a bunch of smaller  $\Delta^q$ , and we can make the process chain homotopy! The standard technique is called the *barycenter subdivision*: you simply divide  $\Delta^q$  from its barycenter according to a certain rule. We give a picture for the barycenter subdivision of  $\Delta^2$  once and twice here:



So, for a given  $\mathcal{U}$ , you can use the Lebesgue number lemma to show that if you subdivide enough time for  $\Delta^q$ , then smaller simplexes will eventually be in some  $U_i \in \mathcal{U}$ .

#### 4.2. Mayer-Vietoris sequence.

**Definition 4.4.** For a space  $X$  and two subspaces  $X_1, X_2$  of  $X$ . We say  $\{X_1, X_2\}$  is a Mayer-Vietoris duo if the inclusion of chain complex

$$S_*(X_1) + S_*(X_2) \rightarrow S_*(X_1 \cup X_2)$$

is a quasi-isomorphism, where  $S_*(X_1) + S_*(X_2)$  is the Minkowski sum of sub-chain complexes in  $S_*(X_1 \cup X_2)$ .

**Example 4.5.** By Theorem 4.2, we have if  $X = X_1 \cup X_2$  with  $X = \text{Int}(X_1) \cup \text{Int}(X_2)$  as well. Then  $\{X_1, X_2\}$  is a Mayer-Vietoris duo.

**Theorem 4.6.** If  $\{X_1, X_2\}$  is a Mayer-Vietoris duo, then there exists a long exact sequence

$$\cdots \rightarrow H_q(X_1 \cap X_2) \xrightarrow{s_q} H_q(X_1) \oplus H_q(X_2) \xrightarrow{a_q} H_q(X_1 \cup X_2) \xrightarrow{\partial} H_{q-1}(X_1 \cap X_2) \rightarrow \cdots,$$

which is called the Mayer-Vietoris sequence.

*Proof.* We start by constructing a short exact sequence of complexes. We first write down the following diagram of inclusions

$$\begin{array}{ccc} & X_1 & \\ i_1 \nearrow & & \searrow j_1 \\ X_1 \cap X_2 & & X_1 \cup X_2 \\ i_2 \searrow & & \nearrow j_2 \\ & X_2 & \end{array}$$

Then we set

$$\begin{aligned} s_{\#} : S_*(X_1 \cap X_2) &\rightarrow S_*(X_1) \oplus S_*(X_2), & c &\mapsto (i_{1\#}(c), -i_{2\#}(c)) \\ a_{\#} : S_*(X_1) \oplus S_*(X_2) &\rightarrow S_*(X_1) + S_*(X_2), & (x, y) &\mapsto j_{1\#}(x) + j_{2\#}(y). \end{aligned}$$

Then one can check that the following is a short exact sequence of chain complexes

$$0 \rightarrow S_*(X_1 \cap X_2) \xrightarrow{s_{\#}} S_*(X_1) \oplus S_*(X_2) \xrightarrow{a_{\#}} S_*(X_1) + S_*(X_2) \rightarrow 0.$$

Then we have a long exact sequence

$$\cdots \rightarrow H_q(X_1 \cap X_2) \xrightarrow{H_q(s_{\#})} H_q(X_1) \oplus H_q(X_2) \xrightarrow{H_q(a_{\#})} H_q(S_*(X_1) + S_*(X_2)) \xrightarrow{\partial} H_{q-1}(X_1 \cap X_2) \rightarrow \cdots.$$

Here we denote  $s_q = H_q(s_{\#})$ , but for the next term, we define

$$a_q : H_q(X_1) \oplus H_q(X_2) \xrightarrow{H_q(a_{\#})} H_q(S_*(X_1) + S_*(X_2)) \xrightarrow{\cong} H_q(X_1 \cup X_2)$$

where the last isomorphism follows from that  $\{X_1, X_2\}$  is a Mayer-Vietoris duo. With this notation, we have the Mayer-Vietoris sequence

$$\cdots \rightarrow H_q(X_1 \cap X_2) \xrightarrow{s_q} H_q(X_1) \oplus H_q(X_2) \xrightarrow{a_q} H_q(X_1 \cup X_2) \xrightarrow{\partial} H_{q-1}(X_1 \cap X_2) \rightarrow \cdots. \quad \square$$

**Exercise 4.7.** Show that the reduced singular homology and homology with coefficients also have the Mayer-Vietoris sequence for the Mayer-Vietoris duo if  $X_1$  and  $X_2$  are non-empty.

**Remark 4.8. Important information on some constructions**

(1) Here, we describe the connecting map

$$H_q(X_1 \cup X_2) \xrightarrow{\partial} H_{q-1}(X_1 \cap X_2) :$$

By definition of Mayer-Vietoris duo, for  $z \in H_q(X_1 \cup X_2)$ , there exist a singular chain  $\sigma$  such that  $\sigma = x_1 + x_2$  and  $x_i \in S_q(X_i)$ . Moreover,  $\sigma$  is a cycle, so  $\partial_q \sigma = \partial_q x_1 + \partial_q x_2 = 0$ , and  $\partial_q x_1 = -\partial_q x_2$ . As  $\partial_q x_i \in S_{q-1}(X_i)$ , then the equation  $\partial_q x_1 = -\partial_q x_2$  shows  $\partial_q x_1 = -\partial_q x_2 \in S_{q-1}(X_1 \cap X_2)$ .

We claim that, by certain diagram chasing,

$$\partial z = \partial[\sigma] = [\partial x_1] = -[\partial x_2].$$

(2) Compare to the definition of  $s_{\#}$  and  $a_{\#}$ , we may also define  $s'_{\#} = (i_{1\#}, i_{2\#})$  and  $a'_{\#} = j_{1\#} - j_{2\#}$ . It does not really matter which pair of conventions we take, but we need to fix one during your application, especially when comparing two Mayer-Vietoris sequences (for example, in Exercise 4.10).

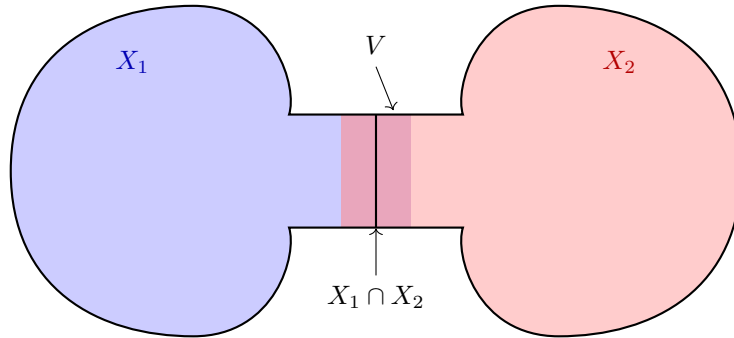
*Supplement material 4.9.* For a topological space  $X$ , the assignment  $U \mapsto S_*(U)$  actually defines a  $\infty$ -cosheaf valued in the derived category of  $\mathbb{Z}$ -modules. The main ingredient to prove the fact is the Mayer-Vietoris sequence. That explains why it really like the definition of sheaf in a proper meaning.

**Exercise 4.10.** Let  $f : X \rightarrow Y$  be a map. Let  $\{X_1, X_2\}$  is a Mayer-Vietoris duo in  $X$  and  $\{Y_1, Y_2\}$  is a Mayer-Vietoris duo in  $Y$  such that  $f(X_i) \subset Y_i$ . Then we have a diagram of Mayer-Vietoris sequences:

$$\begin{array}{ccccccccc} \longrightarrow & H_q(X_1 \cap X_2) & \xrightarrow{s} & H_q(X_1) \oplus H_q(X_2) & \xrightarrow{a} & H_q(X_1 \cup X_2) & \xrightarrow{\partial} & H_{q-1}(X_1 \cap X_2) & \longrightarrow \\ & \downarrow f_q & & \downarrow f_q & & \downarrow f_q & & \downarrow f_q & \\ \longrightarrow & H_q(Y_1 \cap Y_2) & \xrightarrow{s} & H_q(Y_1) \oplus H_q(Y_2) & \xrightarrow{a} & H_q(Y_1 \cup Y_2) & \xrightarrow{\partial} & H_{q-1}(Y_1 \cap Y_2) & \longrightarrow \end{array}$$

To simplify certain discussions, we give one more criterion for the Mayer-Vietoris duo.

**Proposition 4.11.** Let  $X = X_1 \cup X_2$  be a union of two closed subspaces. Assume that  $X_1 \cap X_2$  is a deformation retraction for one open neighborhood  $V$ , then  $\{X_1, X_2\}$  is a Mayer-Vietoris duo.



*Proof.* Let  $V_i = X_i \cup V$ , then we know by the small chain theorem that  $\{V_1, V_2\}$  is a Mayer-Vietoris duo with  $X = V_1 \cup V_2$ .

We have that  $V_i$  deformation retracts to  $X_i$ . In fact, let  $H : V \times I \rightarrow V$  be the deformation retraction homotopy from  $V$  to  $X_1 \cap X_2$ , then we can check that  $H_i : V_i \times V_i$  defined by

$$H_i(x, t) = x, x \in V_i, \quad H_i(x, t) = H(x, t), x \in V \setminus V_i$$

is a deformation retraction from  $V_i$  to  $X_i$ .

Now, we run the proof of Theorem 4.6 to get the following commutative diagram of long exact sequences

$$\begin{array}{ccccccccc} \longrightarrow & H_q(X_1 \cap X_2) & \xrightarrow{s} & H_q(X_1) \oplus H_q(X_2) & \xrightarrow{H_q(a_{\#})} & H_q(S_*(X_1) + S_*(X_2)) & \xrightarrow{\partial} & & \\ & \downarrow & & \downarrow & & \downarrow & & & \\ \longrightarrow & H_q(V = V_1 \cap V_2) & \xrightarrow{s} & H_q(V_1) \oplus H_q(V_2) & \xrightarrow{a} & H_q(X = V_1 \cup V_2) & \xrightarrow{\partial} & & \end{array}$$

In those vertical homomorphisms, we are only unsure if  $H_q(S_*(X_1) + S_*(X_2)) \rightarrow H_q(X)$  is isomorphic or not, and all others are isomorphisms by homotopy invariance. However, in this case, by the five lemma, we conclude that  $H_q(S_*(X_1) + S_*(X_2)) \rightarrow H_q(X)$  is an isomorphism. i.e.,  $\{X_1, X_2\}$  is a Mayer-Vietoris duo.  $\square$

As an application, we can finally compute homology for  $S^n$ .

**Proposition 4.12.** For  $n \geq 1$ , we have  $\tilde{H}_q(S^n) = \mathbb{Z}$  for  $q = n$  and otherwise  $\tilde{H}_q(S^n) = 0$ . Consequently, we have  $H_q(S^n) = \mathbb{Z}$  for  $q = 0, n$  and otherwise  $H_q(S^n) = 0$  by Exercise 2.31, and then  $\chi(S^n) = 1 + (-1)^n$  (recall Definition 3.21).

*Proof.* Consider  $S^n$  and we decompose it into  $D_+^n$  and  $D_-^n$  by upper and lower semi-spheres. Then one can check  $\{D_+^n, D_-^n\}$  is a Mayer-Vietoris duo by Proposition 4.11. Notice that  $D_+^n \cap D_-^n = S^{n-1}$  is the meridian sphere

Then we have the associated Mayer-Vietoris sequence. To simplify the discussion, we use the reduced version:

$$\cdots \rightarrow \tilde{H}_q(D_+^n) \oplus \tilde{H}_q(D_-^n) \rightarrow \tilde{H}_q(S^n) \xrightarrow{\partial} \tilde{H}_{q-1}(S^{n-1}) \rightarrow \tilde{H}_{q-1}(D_+^n) \oplus \tilde{H}_{q-1}(D_-^n) \rightarrow \cdots$$

Now, as  $\tilde{H}_q(D_-^n)$  are trivial for all  $q$ , then we have

$$\tilde{H}_q(S^n) \cong \tilde{H}_{q-1}(S^{n-1}) \cong \tilde{H}_{q-2}(S^{n-2}) \cong \cdots \cong \tilde{H}_{n-q}(S^0).$$

Then we conclude by recall that  $S^0 = \text{pt} \sqcup \text{pt}$  and  $\tilde{H}_q(S^0) = \mathbb{Z}$  when  $q = 0$  and trivial otherwise by the reduced version of Proposition 2.20.  $\square$

We have the following quick application.

**Example 4.13.** Go back to Example 1.4. We have  $f, g : S^1 \rightarrow S^1$  where  $f = \text{id}$  and  $g$  is the constant map.

Then we have  $f$  and  $g$  are not homotopic. In fact,  $f_1 = \text{id}_{\mathbb{Z}}$  and  $g_1 : \mathbb{Z} \rightarrow \mathbb{Z}$  is 0 by Proposition 3.8. So they cannot be homotopic. The point for Proposition 4.12 here is that we now know  $H_1(S^1)$  is non-trivial!

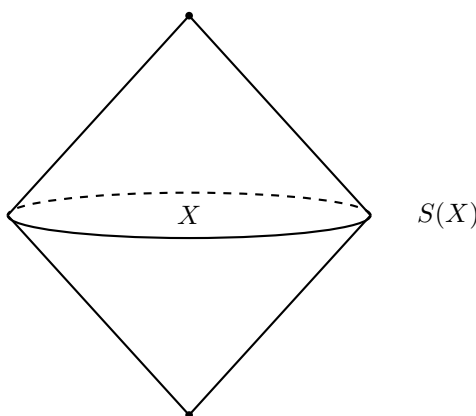
**Exercise 4.14.** Take finitely many topological spaces  $X_i$  with  $i = 1, \dots, d$ , and we consider their wedge sum (recall Definition 1.17)

$$X = X_1 \vee \cdots \vee X_d.$$

1) Show that  $\tilde{H}_q(X) \cong \bigoplus_{i=1}^d \tilde{H}_q(X_i)$ ; 2)\* Try to prove the same result  $X = \bigvee_{i \in I} X_i$  for a infinite index set  $I$ .

Hint: For 1), you may do induction on  $d$ . For 2), you really need to use the universal property of direct sum, as well as a trick you will learn from Exercise 4.36 about compactness of  $\Delta^q$ .

**Exercise 4.15.** Let  $X$  be a space. Its suspension  $S(X)$  is defined as the quotient of  $X \times I$  by collapsing top  $X \times \{1\}$  to one point and the bottom  $X \times \{0\}$  to another point. For example, it is easy to see that  $S(S^{n-1}) \cong S^n$ .



Show that for all  $q$ , we have  $\tilde{H}_q(X) \cong \tilde{H}_{q+1}(S(X))$ . Hint: The subspace of  $SX$  consist of points  $[x, t]$  with  $t \in [0, 1/2]$  is homeomorphic to the cone  $C(X)$ .

### 4.3. Further applications about sphere.

4.3.1. *Brouwer fixed point theorem.* We may also prove the Brouwer fixed point theorem. We start from the following lemma, which is left as an exercise

**Exercise 4.16.** For  $n \geq 2$ , show that for  $i : \partial D^n \cong S^{n-1} \subset D^n$ , there is no continuous map  $r : D^n \rightarrow \partial D^n$  such that  $ri = \text{id}_{\partial D^n}$ . Hint: you shall use  $H_{n-1}(S^{n-1}) = \mathbb{Z}$  and the functoriality of homology.

**Theorem 4.17** (Brouwer fixed point theorem). *For  $n \geq 1$ , every continuous map*

$$f : D^n \rightarrow D^n$$

*has a fixed point.*

*Proof.* Here, we assume  $n \geq 2$ , since the case  $n = 1$  is trivial.

Assume that  $f$  has no fixed point, namely  $f(x) \neq x$  for all  $x \in D^n$ . For each  $x \in D^n$ , define  $r(x)$  to be the unique intersection point of  $\partial D^n$  with the half-line

$$\{f(x) + t(x - f(x)) \mid t \geq 0\}.$$

Then  $r : D^n \rightarrow \partial D^n$  is continuous and satisfies  $ri = \text{id}_{\partial D^n}$  where  $i : \partial D^n \subset D^n$ .

This contradicts Exercise 4.16. Therefore,  $f$  must have a fixed point.  $\square$

4.3.2. *Mapping degree of sphere.* Here, we study further the homology of  $S^n$ . By Proposition 4.12, we know that  $H_n(S^n) \cong \mathbb{Z}$ . We will give an explicit generator and see the effect of the sphere's self-maps.

**Exercise 4.18.** Recall that Exercise 2.18, we construct a cycle  $\sigma_{S^n}$ . Here, we try to show that  $[\sigma_{S^n}]$  is a generator of  $H_n(S^n) \cong \mathbb{Z}$  by the following two steps.

1) Show that under the Mayer-Vietoris boundary map  $\partial : H_n(S^n) \cong H_{n-1}(S^{n-1})$ , we have  $\partial[\sigma_{S^n}] = (-1)^n[\sigma_{S^{n-1}}]$ . 2) The cycle  $\sigma_{S^n}$  can be defined for  $n = 0$ , show that  $[\sigma_{S^0}]$  is a cycle of 2 points that generate  $\tilde{H}_0(S^0) \cong \mathbb{Z}$ .

Comment: Notice, there is a sign shown in 1). To avoid it, one can normalize by using  $\tau_{S^n} = (-1)^{n(n+1)/2}\sigma_{S^n}$ . Then we have  $\partial[\tau_{S^n}] = [\tau_{S^{n-1}}]$ . You might notice that this sign is kind of annoying; this is related to the orientation of  $S^n$  (as a manifold).

You shall also know that we have many other choices of chains that represent the generators.

**Definition 4.19.** Let  $f : S^n \rightarrow S^n$ , and we consider  $f_n : H_n(S^n) \rightarrow H_n(S^n)$ . Since  $H_n(S^n) \cong \mathbb{Z}$ , there exists unique  $d \in \mathbb{Z}$  such that  $f_n(z) = dz$  for all  $h \in H_n(S^n)$ . We define  $d$  as the mapping degree of  $f$ , and denoted by  $\text{deg}(f)$ .

*Remark 4.20.* The notion of mapping degree has now been generalized to many different settings and is deeply rooted in many directions. For example, characteristic class, index theory, enumerative geometry, and many!

We start from some elementary properties.

**Exercise 4.21.** Show the following 1)  $\text{deg}(\text{id}_{S^n}) = 1$ . 2)  $\text{deg}(f \circ g) = \text{deg}(f) \text{deg}(g)$ . 3)  $\text{deg}(\text{const}) = 0$ . 4) If  $f \simeq g$ , then  $\text{deg}(f) = \text{deg}(g)$ . 5) Let  $r : S^n \rightarrow S^n$  be the reflection  $r(x_0, \dots, x_n) = (-x_0, \dots, x_n)$ , we have  $\text{deg}(r) = -1$ . Hint: use  $[\sigma_{S^n}]$  (equivalently  $[\tau_{S^n}]$ ) or do induction using  $\partial : H_n(S^n) \cong H_{n-1}(S^{n-1})$ .

Next, we show the ampleness of the degree for the sphere.

**Exercise 4.22.** We show that for all  $n \geq 1$  and  $d \in \mathbb{Z}$ , there exists a map  $f_d : S^n \rightarrow S^n$  with degree  $d$ .

Hint: You can start from  $n = 1$ , where  $f(z) = z^d$  is the candidate. Then you can do induction using suspension construction Exercise 4.15, which is functorial both at the space level and at the homology group level.

By those two exercises, we can see that

$$\text{deg} : [S^n, S^n] \rightarrow \mathbb{Z}$$

is a surjective homomorphism between multiplicative monoid (recall that  $[X, X]$  is the monoid of homotopy class of maps between  $X$ ).

The following is harder, and we omit its proof.

**Theorem 4.23** (Hopf). *The degree map  $\text{deg} : [S^n, S^n] \rightarrow \mathbb{Z}$  is injective.*

*Supplement material 4.24.* Now, you can use the tool from this section to show the Jordan curve theorem and the invariance of domain. We refer to [Hat02, Section 2.B] for further reading.

**4.4. Mapping cone sequence.** Here, we study the mapping cone using the Mayer-Vietoris sequence.

We start from some topological constructions.

Let  $X$  be a space, and  $A \subset X$  is a subspace. We take  $f : A \rightarrow Y$ , and then we define

$$Y \cup_f X := X \sqcup Y / (a \in A) \sim (f(a) \in Y),$$

whose topology is the quotient topology inherited from the disjoint union  $X \sqcup Y$ .

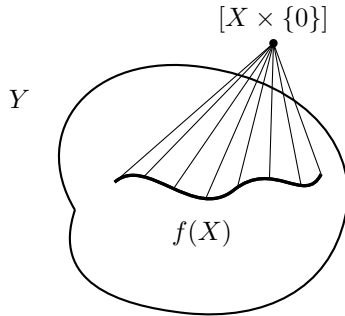
**Example 4.25.** Let  $(X, A) = (D^n, S^{n-1})$ , and  $f : S^{n-1} \rightarrow Y$  be a map. Then we call  $Y \cup_f D^n$  as the space glued with an  $n$ -dimensional cell  $D^n$  on  $Y$ .

**Example 4.26.** Let  $f : X \rightarrow Y$  be a map and consider the pair  $(C(X), X \times 1)$  where we regard the bottle  $X \times 1$  as  $X$ . Then we define the (topological) mapping cone as

$$C(f) = Y \cup_f C(X).$$

The inclusion  $Y \rightarrow Y \cup_f C(X)$  induces a continuous map  $e : Y \rightarrow C(f)$

Notice the following property:  $f \mapsto C(f)$  is functorial. If  $f \simeq g$ , then  $C(f) \simeq C(g)$ .



$$C(f) = (Y \cup_f C(X)) / ([x, 1] \sim f(x))$$

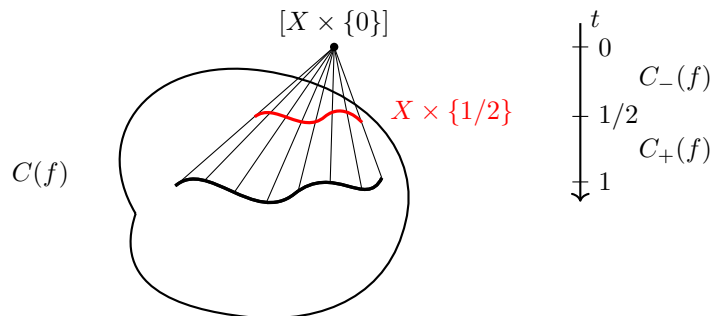
**Proposition 4.27.** *Let  $f : X \rightarrow Y$  be a map, then there exists a long exact sequence*

$$\cdots \rightarrow \tilde{H}_q(X) \xrightarrow{f_q} \tilde{H}_q(Y) \xrightarrow{e_q} \tilde{H}_q(C(f)) \xrightarrow{\partial} \tilde{H}_{q-1}(X) \rightarrow \cdots,$$

where  $e : Y \rightarrow C(f)$  is the inclusion map.

*Proof.* We apply the Mayer-Vietoris sequence to the Mayer-Vietoris duo  $\{C_+(f), C_-(f)\}$  in  $C(f)$  given by

$$C_+(f) = Y \cup_f X \times [1/2, 1], \quad C_-(f) = X \times [0, 1/2] / X \times 0.$$



Notice that  $C_-(f) \cong C(X)$  is contractible, then  $\tilde{H}_q(C_-(f)) = 0$  for all  $q$ . So, the reduced Mayer-Vietoris sequence becomes

$$\cdots \rightarrow \tilde{H}_q(X \times 1/2) \rightarrow \tilde{H}_q(C_+(f)) \rightarrow \tilde{H}_q(C(f)) \xrightarrow{\partial} \tilde{H}_{q-1}(X \times 1/2) \rightarrow \cdots .$$

We also notice that  $C_+(f) \simeq Y$ . Then we conclude by replacing  $\tilde{H}_q(C_+(f))$  by  $\tilde{H}_q(Y)$  and carefully match related maps  $\square$

**Exercise 4.28.** Finish the proof of Proposition 4.27. And prove that for the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \alpha & & \downarrow \beta \\ X' & \xrightarrow{f'} & Y' \end{array}$$

there exists a map  $\gamma : C(f) \rightarrow C(f')$  and the following commutative diagram of mapping cone sequences:

$$\begin{array}{ccccccccc} \longrightarrow & \tilde{H}_q(X) & \xrightarrow{f_q} & \tilde{H}_q(Y) & \xrightarrow{e_q} & \tilde{H}_q(C(f)) & \xrightarrow{\partial} & \tilde{H}_{q-1}(X) & \longrightarrow \\ & \downarrow \alpha_q & & \downarrow \beta_q & & \downarrow \gamma_q & & \downarrow \alpha_{q-1} & \\ \longrightarrow & \tilde{H}_q(X') & \xrightarrow{f'_q} & \tilde{H}_q(Y') & \xrightarrow{e'_q} & \tilde{H}_q(C(f')) & \xrightarrow{\partial} & \tilde{H}_{q-1}(X') & \longrightarrow . \end{array}$$

*Remark 4.29.* If you want to use the non-reduced homology in mapping cone sequence, it is clear that nothing will change for  $q \geq 1$ , but for  $q = 0$ , the sequence will be

$$\cdots \rightarrow H_1(C(f)) \xrightarrow{\partial} H_0(X) \xrightarrow{f_0 \oplus \varepsilon_X} H_0(Y) \oplus \mathbb{Z} \xrightarrow{E} H_0(C(f)) \rightarrow 0,$$

where  $E(b, n) = e_0(b) - n[v]$  for arbitrary but a fixed  $v \in C(f)$ . We use the same  $\partial$  since it factor through the (non-canonical) inclusion  $\tilde{H}_0(X) \hookrightarrow H_0(X)$  by the reduced version of  $\partial$ . More concretely, for  $p, q \in X \simeq X \times 1/2$  and pick paths  $\alpha \subset C_+(f)$  (resp.  $\beta \subset C_-(f)$ ) from  $p$  to  $q$  (resp. from  $q$  to  $p$ ), then  $c = \alpha + \beta \in S_1(C(f))$  and  $\partial[c] = \partial[\alpha] = [q] - [p]$ . It is clear that  $\partial[c]$  is a class in the reduced homology no matter which base point we take.

We left the verification to readers.

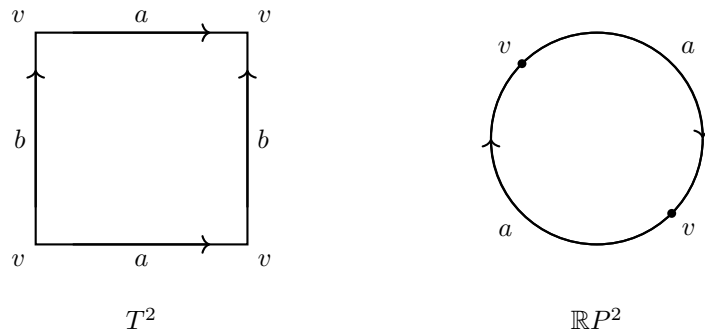
*Remark 4.30.* This construction essentially motivates the definition of the algebraic mapping cone. In fact, by a careful construction, you can see from the chain level Mayer-Vietoris sequence how to give the algebraic mapping cone.

**Corollary 4.31.** Let  $f : S^{n-1} \rightarrow X$  be a map, we have 1)  $\tilde{H}_q(X \cup_f D^n) = \tilde{H}_q(X)$  for  $q \neq n, n-1$ ; 2) an exact sequence

$$0 \rightarrow \tilde{H}_n(X) \rightarrow \tilde{H}_n(X \cup_f D^n) \xrightarrow{\partial} \tilde{H}_{n-1}(S^{n-1}) \simeq \mathbb{Z} \xrightarrow{f_{n-1}} \tilde{H}_{n-1}(X) \rightarrow \tilde{H}_{n-1}(X \cup_f D^n) \rightarrow 0.$$

*Proof.* Notice that  $D^n \cong C(S^{n-1})$ . Then it follows from Proposition 4.27.  $\square$

**Example 4.32.** We now compute  $\tilde{H}_n(T^2)$ . We may treat  $T^2$  as the quotient of the square  $I^2$  in the following manner by identifying edges with the same labeling.



We take  $X = S^1 \vee S^1$ , i.e., the boundary of the square after the quotient (where the wedge point is  $v$ ), and  $f : S^1 \rightarrow S^1 \vee S^1$  is defined as  $f : S^1 \cong \partial I^2 \xrightarrow{q} S^1 \vee S^1$  where  $q$  is the quotient map. Then we have that  $T^2 \cong X \cup_f D^2$  (here, we think of the closed square  $I^2$  as  $D^2$ .)

We know that  $\tilde{H}_q(S^1 \vee S^1) = \mathbb{Z}^2$  only when  $q = 1$  and otherwise 0 by Exercise 4.14.

By Corollary 4.31, we know  $\tilde{H}_0(T^2) = \tilde{H}_0(S^1 \vee S^1) = 0$ , and we have the following exact sequence

$$0 \rightarrow \tilde{H}_2(T^2) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{f_1} \mathbb{Z}^2 \rightarrow \tilde{H}_1(T^2) \rightarrow 0.$$

So, we need to compute  $H_1(S^1) = \mathbb{Z} \xrightarrow{f_1} \mathbb{Z}^2 = H_1(S^1 \vee S^1)$ . We keep track of the proof of Exercise 4.14, we know the projection of  $f_1$  into the first factor of  $\mathbb{Z}$  is given by the degree of  $S^1$  along the first circle in the wedge  $S^1 \vee S^1$  (i.e.  $a$  in the picture), which is 0: go along  $a$  circle once and back (i.e. go along  $a$  with the opposite direction) is homotopic to the constant map. Similarly, for the second wedge component. Therefore, we have  $f_1 = 0$ .

Consequently, by the exact sequence above, we have

$$\tilde{H}_2(T^2) \cong \mathbb{Z}, \quad \tilde{H}_1(T^2) \cong \mathbb{Z}^2,$$

and two generators of  $\tilde{H}_1(T^2)$  are represent by circles gives meridian and longitude of  $T^2$ .

**Example 4.33.** Similarly, the real projective plane  $\mathbb{R}P^2 = S^1 \cup_f D^2$  can be glued along  $f : S^1 \rightarrow S^1$  with  $f(z) = z^2$  of degree 2. It is clear that  $\mathbb{R}P^2$  is path-connected, so  $H_0 = \mathbb{Z}$ .

We then write down the mapping cone sequence

$$0 \rightarrow \tilde{H}_2(S^1) \rightarrow \tilde{H}_2(\mathbb{R}P^2) \xrightarrow{\partial} \tilde{H}_1(S^1) \simeq \mathbb{Z} \xrightarrow{f_1} \tilde{H}_1(S^1) \rightarrow \tilde{H}_1(\mathbb{R}P^2) \rightarrow 0,$$

i.e.

$$0 \rightarrow \tilde{H}_2(\mathbb{R}P^2) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \tilde{H}_1(\mathbb{R}P^2) \rightarrow 0.$$

Then we have  $H_2(\mathbb{R}P^2) = 0$  and  $H_1(\mathbb{R}P^2) = \mathbb{Z}/2$ .

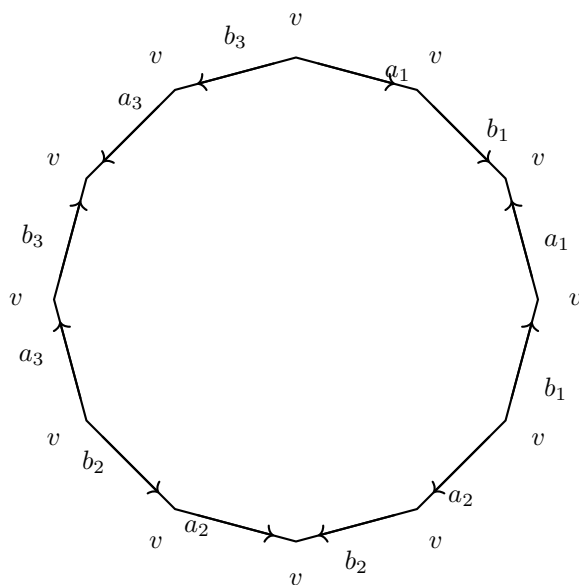
It is interesting to consider the  $\mathbb{F}_2$  coefficient. In this case, the exact sequence becomes

$$0 \rightarrow \tilde{H}_2(\mathbb{R}P^2; \mathbb{F}_2) \xrightarrow{\partial} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \rightarrow \tilde{H}_1(\mathbb{R}P^2; \mathbb{F}_2) \rightarrow 0,$$

and then  $H_2(\mathbb{R}P^2; \mathbb{F}_2) = \mathbb{F}_2$  and  $H_1(\mathbb{R}P^2; \mathbb{F}_2) = \mathbb{F}_2$ . (You can also get the  $\mathbb{F}_2$  computation from the universal coefficient theorem Theorem 2.28).

It is our first example that the coefficient of homology really makes a difference.

**Exercise 4.34.** Let  $\Sigma$  be a genus  $g$  closed surface,  $\Sigma$  can be constructed by gluing a  $4g$ -gon like  $T^2$  (this is the content for the classification theorem for closed surfaces).



Show that

$$\tilde{H}_2(\Sigma) \cong \mathbb{Z}, \quad \tilde{H}_1(\Sigma) \cong \mathbb{Z}^{2g}, \quad \chi(\Sigma) = 2 - 2g.$$

**Exercise 4.35.** We study complex projective space here. We consider  $S^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid |z| = 1\}$  and  $S^1$  acts on it by  $e^{i\theta} \cdot z = e^{i\theta} z$ . We define  $\mathbb{C}P^n$  as the quotient space  $S^{2n+1}/S^1$ .

Show that 1)  $\mathbb{C}P^{n+1} \cong \mathbb{C}P^n \cup_{\pi_n} D^{2n+2}$  for the quotient map  $\pi_n : S^{2n+1} \rightarrow \mathbb{C}P^n = S^{2n+1}/S^1$ . 2)  $H_q(\mathbb{C}P^n) = \mathbb{Z}$  for  $q = 0, 2, \dots, 2n$  and trivial otherwise. 3) Let  $q < 2n + 1$ , then the inclusion  $\mathbb{C}P^n \subset \mathbb{C}P^{n+1}$  induces isomorphism  $H_q(\mathbb{C}P^n) \subset H_q(\mathbb{C}P^{n+1})$ .

Hint: Notice that  $\mathbb{C}P^1 \cong S^2$  (the projection  $\pi_1 : S^3 \rightarrow \mathbb{C}P^1 \cong S^2$  is called the hopf fibration). Then you can prove by induction from this case.

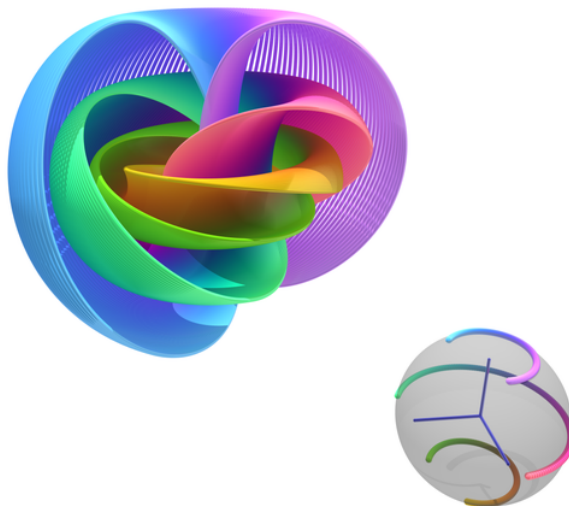


FIGURE 1. Hopf fibration. From Wikipedia.

**Exercise 4.36.** Now, consider  $\mathbb{C}P^\infty$ , which is defined as  $S^\infty/S^1$ , where  $S^\infty$  is defined in Exercise 1.18. By the inclusion  $S^{2n+1} \subset S^\infty$ , we modulo it by  $S^1$ -action, and then we have an inclusion  $i_N : \mathbb{C}P^N \subset \mathbb{C}P^\infty$  with  $\mathbb{C}P^\infty = \bigcup_N \mathbb{C}P^N$ . Show that for all  $q \geq 0$ , there exists  $N > q$ , we have  $H_q(\mathbb{C}P^N) \xrightarrow{\cong} H_q(\mathbb{C}P^\infty)$ . Consequently, we have  $H_q(\mathbb{C}P^\infty) \cong \mathbb{Z}$  if  $q = 2k$  for  $k \geq 0$ , and otherwise trivial.

Hint: You may use the following idea: For a singular simplex  $\sigma : \Delta^q \rightarrow \mathbb{C}P^\infty$ , as  $\Delta^q$  is compact, its image is also compact, so there exists  $N$  such that the image of  $\sigma$  is in  $\mathbb{C}P^N \subset \mathbb{C}P^\infty$ . This is enough to show  $H_q(i_N)$  is surjective. To say injectivity, you may use the same idea and Exercise 4.35-(3).

**Exercise 4.37.** \* We have do  $\mathbb{R}P^2$  in Example 4.33, now you may try  $\mathbb{R}P^n$ . As the case of  $\mathbb{R}P^2$ , you should be careful on the coefficient.

At the end, we use mapping cone to prove the invariance for dimension

**Proposition 4.38.** Let  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^m$  are open sets. Suppose  $\varphi : U \rightarrow V$  is a homeomorphism, then  $m = n$ .

*Proof.* We pick a closed ball in  $i : D \subset U$  ( $D \cong D^n$ ), this is possible by taking an inner closed ball inside an open ball. We also pick  $x \in D$  (for example the center of  $D$ )

Consider  $j_x : U \setminus x \hookrightarrow U$  and  $d_x : D \setminus x \hookrightarrow D$ .

Then we have  $i$  induces homotopy equivalence

$$S^n \simeq C(d_x) \xrightarrow{\cong} C(j_x).$$

Therefore, we have

$$\tilde{H}_n(C(d_x)) \cong \mathbb{Z}, \quad \tilde{H}_n(C(j_x)) \cong 0, \quad q \neq n.$$

We can apply the same discussion to  $V$ .

Now, if  $U$  and  $V$  are homeomorphism, we must have

$$\varphi_q : H_q(C(j_x)) \cong H_q(C(j_{\varphi(x)})).$$

Then we must have  $n = m$  since they must be non-zero for the same  $q$ .  $\square$

*Remark 4.39.* Here, the construction  $H_q(C(X \setminus x \hookrightarrow X))$  is called **local homology** of  $X$  near  $x$ . It would be very useful when we discuss manifolds. The above argument already shows that  $X$  is a manifold, then  $H_n(C(X \setminus x \hookrightarrow X))$  is trivial otherwise.

*Supplement material 4.40.* Traditionally, there exists an equivalent toolkit for computation: relative homology + excision principle, where we will present in Section 9.

However, we choose this slightly different way to build the theory entirely based on Mayer-Vietoris and mapping cone.

One reason for that: this flavor is more close to sheaf theory and stable  $\infty$ -categories (a modern version of triangulated categories), where relative homology is either tricky to define, or simply defined using mapping cone.

It makes relative homology less useful for certain situations (but NOT for all!). To save some time, we left the relative homology as a reading section at the end of the notes.

## 5. CELLULAR HOMOLOGY

Generally, Mayer-Vietoris sequence and mapping cone sequence are effective tools for computation. But still need some case-by-case studies. Here, we define a class of spaces and a certain homology theory such that: the chains are not too huge, the differential is computable in finite time, and the resulting homology coincides with singular homology. Prototype of the construction is the discussion of Subsection 4.4.

**Warning:** The entire section would serve as an exercise sheet.

**5.1. CW complex.** Recall that  $D^n$  is the closed disk, and we write  $E^n = \text{Int}(D^n)$ . They are called cells in some content.

**Definition 5.1.** For a topological space  $X$ , we say a CW decomposition or cellular decomposition consists of the following data

(1) A filtration of closed subspaces

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \dots \subset X^q \subset \dots, \quad X = \bigcup_q X^q,$$

where  $X^q$  is called the  $q$ -skeleton of  $X$ .

(2)  $X^q$  is glued from  $X^{q-1}$  by  $q$ -cells. Precisely:

For each  $q \geq 0$ , there exists a family of continuous maps, called the characteristic maps of the  $q$ -cells,

$$\varphi_\alpha^q : D^q \rightarrow X^q, \quad \alpha \in I_q,$$

such that

- $\varphi_\alpha^q(\partial D^q) \subset X^{q-1}$  for all  $\alpha \in I_q$ ,
- the restriction on  $E^q = \text{Int}(D^q)$

$$\varphi_\alpha^q|_{E^q} : E^q \rightarrow X^q \setminus X^{q-1}$$

is a homeomorphism onto its image, and

- if we set  $\varphi^q := \sqcup_{\alpha \in I_q} \varphi_\alpha^q : \bigsqcup_{\alpha \in I_q} D^q \rightarrow X^q$ , then

$$X^q \cong X^{q-1} \cup_{\varphi^q} \bigsqcup_{\alpha \in I_q} D^q.$$

The image  $e_\alpha^q := \varphi_\alpha^q(E^q)$  is called a  $q$ -cell of  $X$ . We also call  $\partial\varphi_\alpha^q = \varphi_\alpha^q|_{S^{q-1}}$  the gluing map of the cell  $e_\alpha^q$ .

(3) The topology of  $X$  is homeomorphic to the weak topology:  $F \subset X$  is closed if and only if  $F \cap X^q$  are closed for all  $q$ .

A CW complex or cellular complex is a topological space with a fixed CW decomposition. We say  $X$  is a finite-type CW complex if there are only finitely many cells in each dimension  $q$ , and say  $X$  is a finite CW complex if the total number of all cells are finite.

*Remark 5.2.* (1) CW means Closure finite and Weak topology. The notion was introduced by J. H. C. Whitehead, which is a funny coincidence (maybe intentional) that CW appears again.

(2) **Warning:** When you first learn all the machines here, you may consider finite-type CW complexes only to avoid too much technical discussion, especially about the weak topology in the absence of infinitely many  $q$ -cells. But most of the results hold without finiteness, although the proof is harder (the only exception concerns the product of CW complexes).

(3) By definition,  $X^0$  only consists of some isolated points, and  $X^q$  has at most  $q$ -cells.

(4) One can prove that: CW complexes are Hausdorff; finite CW complexes are compact; finite-type CW complexes are locally compact.

**Example 5.3.** (1) The sphere  $S^n$  is a CW complex with only one 0-cell and one  $n$ -cell.

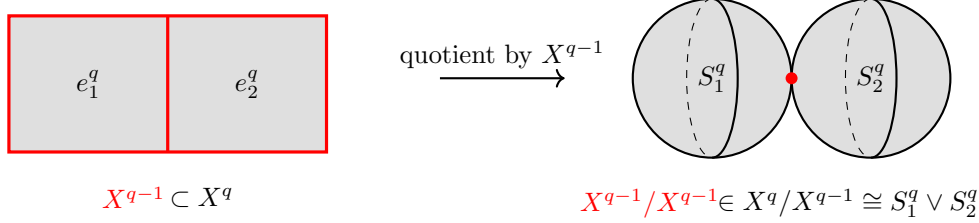
(2) In Example 4.32, we give  $T^2$  a CW decomposition with one 0-cell  $v$ , two 1-cells  $a, b$  and one 2-cell  $I^2$ . The 1-skeleton is  $S^1 \vee S^1$ .

(3) You can see that  $\mathbb{R} = \cup_n [n, n+1]$  can be regarded as a non finite-type CW complex. Also,  $\mathbb{C}P^\infty$  is a finite-type but not finite CW complex.

**Exercise 5.4.** The exercise provides certain preparation for our applications later.

Let  $X$  be a CW complex, show that all characteristic maps of  $q$ -cells  $\varphi_\alpha^q$  combined together induce a homeomorphism between pointed space

$$\Phi : \left(\bigvee_{\alpha} S_{\alpha}^q, \text{pt}\right) \xrightarrow{\cong} (X^q/X^{q-1}, X^{q-1}/X^{q-1}).$$



Hint: By definition of  $\varphi_\alpha^q$ , it descends to  $\Phi_\alpha^q : S^q = D^q/\partial D^q \rightarrow X^q/X^{q-1}$ . Then all of them form a map  $\sqcup_\alpha \Phi_\alpha^q : \sqcup S_\alpha^q \rightarrow X^q/X^{q-1}$ . Then show  $\sqcup_\alpha \Phi_\alpha^q$  descends to the wedge sum to get  $\Phi$ , which matches the marked points. Then by definition, you may easily show  $\Phi$  is a continuous bijection. Then you conclude  $\Phi$  is a homeomorphism: If  $X$  is finite-type, you can argue from a short-cut that all spaces here are compact Hausdorff spaces (notice that  $X^q$  is a finite CW complex); in the general case, you may use the following point set topology result: if  $q_1 : Y \rightarrow Z_1$  and  $q_2 : Y \rightarrow Z_2$  are two quotient map (i.e.,  $q_i$  are surjective continuous map and  $Z_i$  equip the quotient topology), and  $q_1, q_2$  define the same equivalence relation, then there exists a homeomorphism between  $Z_1$  and  $Z_2$ .

A class of CW complexes, called  $\Delta$ -complex, is more combinatorial and easier to understand. It also deeply relates to the so-called simplicial complex (an ancestor of CW complex), but not exactly the same.

**Definition 5.5.** A  $\Delta$ -decomposition (or  $\Delta$ -complex structure) of  $X$  is CW decomposition of  $X$  such that

- All characteristic maps are singular simplexes  $\varphi_\alpha^q : \Delta^q \rightarrow X$ . (Here, you may feel weird since  $\Delta^q \cong D^q$ , the point here is we do not want to specify those homeomorphisms  $\Delta^q \cong D^q$ , which makes the following condition more natural.)
- For all  $\alpha \in I_q$  and  $0 \leq i \leq q$ , there exists  $\beta \in I_{q-1}$  such that the composition  $\Delta^{q-1} \xrightarrow{\delta_q^i} \Delta^q \xrightarrow{\varphi_\alpha} X$  equals  $\varphi_\beta^{q-1}$ .

**Exercise 5.6.** In Example 2.8, we define some singular simplexes  $\sigma_I$ . Show that all those  $\sigma_I$  and their all faces (including faces of faces, and so on) define a  $\Delta$ -complex structure on  $S^n$ .

Another example of  $\Delta$ -complex structure is for  $T^2$ , where we present in Example 8.34.

**5.2. Cellular homology.** Here, we define a particular chain complex for CW complex, say, the cellular chain. And we eventually show that the homology for the cellular chain computes the singular homology for CW complexes.

Again, we may assume  $X$  is a finite-type CW complex to avoid more technical details for your first reading. But all results and constructions in this section are true for general CW complexes.

Let us take a CW complex  $X$ , for<sup>1</sup>  $k \geq 0$ , we define spaces

$$Q^k = X^k/X^{k-1}, \quad E^k = X^k/X^{k-2},$$

where  $X^{-1} = X^{-2} := \emptyset$ , together with the evident commutative diagram for  $k \geq 1$

$$\begin{array}{ccccc} X^{k-1} & \xrightarrow{i_k} & X^k & \xrightarrow{u_k} & C(i_k) \\ \downarrow q_{k-1} & & \downarrow \pi_k & & \downarrow c_k \\ Q^{k-1} & \xrightarrow{j_k} & E^k & \xrightarrow{v_k} & C(j_k) \end{array}$$

<sup>1</sup>In most of subsections, we use  $q$  to indicate grading; the only exception is this subsection where we prefer  $k$ .

where  $u_k, v_k$  are natural maps comes with mapping cone,  $c_k$  is induced by  $\pi_k$  on  $X^k/\sim$  and induced by  $q_{k-1}$  on  $C(X^{k-1})/\sim$  (the map  $\gamma$  in Exercise 4.28). Evidently, we set  $C(i_0) = C(j_0) = X^0$  and  $q_0 = \pi_0 = c_0 = u_0 = v_0 = \text{id}$ .

*Remark 5.7.* The motivation for the construction is that: Based on the experience of Corollary 4.31, only  $H_k, H_{k-1}$  changes if we glue some  $k$ -cell. So, we would like to focus on  $X^k, X^{k-1}$  by modulo  $X^{k-2}$  on the space level. You might see that  $j_k$  could be understood as  $i_k/X^{k-2}$  in some sense.

In the next exercise, we want to give some (space level) models for mapping cones. Precisely, we will see that both  $C(i_k)$  and  $C(j_k)$  has the homotopy type of  $Q^k$ . Moreover, you may think of  $u_k$  and  $v_k$  as quotient maps  $X^k \rightarrow Q^k$  or  $E^k \rightarrow Q^k$  up to homotopy. To make sure compatibility of this change of point of view, we also define some maps and explain some relations between them.

**Exercise 5.8** (Definition for cellular chain groups). For  $k \geq 1$ , we define the following maps

$$s_k : C(i_k) \rightarrow Q^k, \quad t_k : C(j_k) \rightarrow Q^k,$$

where  $s_k$  sends  $[y] \in X^k/\sim \subset C(i_k)$  to  $[y] \in X^k/X^{k-1}$  and sends  $[y] \in C(X^{k-1})/\sim \subset C(i_k)$  to  $[X^{k-1}] \in X^k/X^{k-1}$ , and  $t_k$  is defined in a similar way that sends  $[y] \in E^k/\sim$  to  $[y]$  under the quotient map  $E^k \rightarrow Q^k$  and sends  $C(X^{k-1})/\sim$  to  $[X^{k-1}] \in X^k/X^{k-1}$ ; and

$$a_k : Q^k \rightarrow C(i_k), \quad b_k : Q^k \rightarrow C(j_k),$$

where  $a_k$  send  $X^k/X^{k-1}$  into  $X^k/\sim \subset C(i_k)$  and  $b_k$  sends  $X^k/X^{k-1}$  into  $E^k/\sim \subset C(j_k)$  via  $\pi_k$ . You can check the well-definedness directly.

Show that  $s_k$  and  $a_k$  (resp.  $t_k$  and  $b_k$ ) are homotopy inverse of each other with  $s_k = t_k \circ c_k$ ,  $s_k \circ a_k = \text{id}$ ,  $t_k \circ b_k = \text{id}$  (you need to construct  $a_k \circ s_k \simeq \text{id}$ ,  $b_k \circ t_k \simeq \text{id}$ ).

Remark that, as before, we simply set  $s_0 = t_0 = a_0 = b_0 = \text{id}$ .

Then we have a commutative diagram of ISOMORPHISMS

$$\begin{array}{ccc} H_q(C(i_k)) & \xrightarrow{H_q(s_k)} & H_q(Q^k) \\ \downarrow H_q(c_k) & & \parallel \\ H_q(C(j_k)) & \xrightarrow{H_q(t_k)} & H_q(Q^k). \end{array}$$

And as we already show Exercise 5.4 that  $Q^k \cong \vee_{\alpha} S_{\alpha}^k$ , we have  $k = q$  all four terms are isomorphic to  $\oplus_{\alpha} \mathbb{Z}[e_{\alpha}^k]$  for  $k \geq 1$  by Exercise 4.14 and for  $k = 0$  by Proposition 2.20.

**Notice** that here we use an annoying notation  $H_k(f_k)$ . Later we will simply write  $f_k$  if no further confusion.

Then we **define** for  $k \geq 0$

$$C_k(X) := H_k(Q^k) \cong \oplus_{\alpha} \mathbb{Z}[e_{\alpha}^k],$$

and  $C_k(X) = 0$  for negative  $k$ .

**Exercise 5.9** (Definition for cellular differential). For  $k \geq 1$ , show that

(1) We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_k(X^{k-1}) & \xrightarrow{i_k} & H_k(X^k) & \xrightarrow{q_k} & C_k(X) & \xrightarrow{\partial} & H_{k-1}(X^{k-1}) & \longrightarrow & H_{k-1}(X^k) \\ \parallel & & \downarrow & & \downarrow \pi_k & & \parallel & & \downarrow q_{k-1} & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & H_k(E^k) & \xrightarrow{\pi_k} & C_k(X) & \xrightarrow{d_k} & C_{k-1}(X) & \longrightarrow & H_{k-1}(E^k) \end{array}$$

where two horizontal sequences are exact.

**Warning:** Be careful, here we really want to use the non-reduced version of mapping cone sequence. So, in the case  $k = 1$ , we replace the right most terms by  $H_0(E^1) \oplus \mathbb{Z} = H_0(X^1) \oplus \mathbb{Z}$  as explained in Remark 4.29.

Hint: To construct  $\partial$  and exact sequences, we first use the mapping cone sequence Proposition 4.27 (but the non-reduced version in low degrees, compare Remark 4.29), and then use Exercise 5.8 to show that maps in mapping cone sequence can be replaced by maps here. The commutativity relations for maps is then either true on spaces level, or by commutative relations in Exercise 5.8. We also use Exercise 5.8 to show some homologies are trivial.

In particular, we have a homomorphism

$$d_k : C_k(X) \rightarrow C_{k-1}(X).$$

(2) Show  $d_k d_{k+1} = 0$  by embedding the composition  $d_k d_{k+1}$  into the commutative diagram

$$\begin{array}{ccccccc}
 & & C_{k+1}(X) & & & & \\
 & & \downarrow \partial & \searrow d_{k+1} & & & \\
 & \longrightarrow & H_k(X^k) & \xrightarrow{q_k} & C_k(X) & \xrightarrow{\partial} & H_{k-1}(X^{k-1}) \longrightarrow \\
 & & & & \searrow d_k & & \downarrow q_{k-1} \\
 & & & & & & C_{k-1}(X),
 \end{array}$$

where the horizontal line is the mapping cone sequence for  $i_k$ .

**Definition 5.10.** For a CW complex  $X$ , we define its cellular chain complex by

$$C_*(X) = (C_k(X), d_k).$$

Its homology groups are called the cellular homology of  $X$ , denoted by  $H_k^{CW}(X)$ .

**Theorem 5.11.** We have  $H_k^{CW}(X) \cong H_k(X)$  for a finite CW complex. In particular, cellular homology is independent of choices of CW decompositions.

We prove it through the following 2 exercises.

**Exercise 5.12.** For a CW complex  $X$ , show that 1)  $H_k(X^n) = 0$  for  $k > n$ . 2)  $H_k(X^n) \cong H_k(X^{n+1})$  for  $k < n$ . 3)  $H_k(X^n) \cong H_k(X)$  for  $k < n$ . Hint: Use the mapping cone sequence for  $i_k$  for 1) and 2); and for 3) you shall use the idea in Exercise 4.36.

**Exercise 5.13.** Using the commutative diagram in Exercise 5.9, do some diagram chasing to show that 1)  $\ker d_k \cong H_k(X^k)$ , and  $\text{im}(d_{k+1}) \cong \ker(H_k(X^k) \xrightarrow{H_k(i_{k+1})} H_k(X^{k+1}))$ . 2) Conclude the result of Theorem 5.11 using Exercise 5.12 - 2) and 3).

**Exercise 5.14** (Explicit formula for cellular differential). Let  $X$  be a CW complex. Recall  $C_k(X) = \bigoplus_{\alpha} \mathbb{Z}[e_{\alpha}^k]$  where  $e_{\alpha}^k$  are  $k$ -cells.

For each pair  $(e_{\alpha}^k, e_{\beta}^{k-1})$  with  $k \geq 2$ , define

$$f_{\alpha\beta} : S_{\alpha}^{k-1} \xrightarrow{\partial\varphi_{\alpha}^k} X^{k-1} \xrightarrow{q_{k-1}} Q^{k-1} = X^{k-1}/X^{k-2} \xrightarrow{p_{\beta}} S_{\beta}^{k-1}$$

to be the composite, where the last map  $p_{\beta}$  is the projection to the  $\beta$ -th wedge summand

$$X^{k-1}/X^{k-2} \cong \bigvee_{\beta} S_{\beta}^{k-1}.$$

In this exercise, we shall show that for  $k \geq 2$

$$d_k([e_{\alpha}^k]) = \sum_{\beta} \text{deg}(f_{\alpha\beta})[e_{\beta}^{k-1}],$$

i.e. the matrix coefficient of  $d_k$  from  $e_{\alpha}^k$  to  $e_{\beta}^{k-1}$  is exactly  $\text{deg}(f_{\alpha\beta})$ .

(1) Let  $u_k \in H_k(S^k) \cong \mathbb{Z}$  be a generator (in fact, we prefer  $u_k = [\tau_{S^k}]$  in Exercise 4.18 here.), and recall the map  $\Phi_{\alpha}^k : S^k \cong D^k/S^{k-1} \rightarrow Q^k = X^k/X^{k-1}$ . Show that  $[e_{\alpha}^k] = (\Phi_{\alpha}^k)_k(u_k) \in H_k(Q^k) = C_k(X)$ .

(2) By definition of characteristic map  $\varphi_\alpha^k$ , we have the following commutative diagram of maps

$$\begin{array}{ccccccc} S^{k-1} & \xrightarrow{i} & D^k & \longrightarrow & D^k/S^{k-1} \cong S^k & \xleftarrow{\cong} & C(i) \\ \downarrow q_{k-1} \circ \partial \varphi_\alpha^k & & \downarrow \pi_k \circ \varphi_\alpha^k & & \downarrow \Phi_\alpha^k & & \downarrow c_\alpha^k \\ Q^{k-1} & \xrightarrow{j_k} & E^k & \longrightarrow & Q^k & \xleftarrow[\simeq]{t_k} & C(j_k), \end{array}$$

where  $c_\alpha^k$  is the map between cones constructed in Exercise 4.28 (i.e.  $\gamma$  there).

Prove the following commutative diagram of homologies follows from Exercise 4.28:

$$\begin{array}{ccc} H_k(S^k) & \xrightarrow{\partial} & H_{k-1}(S^{k-1}) \\ (\Phi_\alpha^k)_k \downarrow & & \downarrow (q_{k-1} \circ \partial \varphi_\alpha^k)_{k-1} \\ C_k(X) = H_k(Q^k) & \xrightarrow{d_k} & H_{k-1}(Q^{k-1}) = C_{k-1}(X), \end{array}$$

and show that

$$d_k([e_\alpha^k]) = d_k((\Phi_\alpha^k)_k(u_k)) = (q_{k-1} \circ \partial \varphi_\alpha^k)_{k-1}(\partial(u_k)).$$

(3) Show that  $\partial(u_k)$  is a generator of  $H_{k-1}(S^{k-1}) \cong \mathbb{Z}$ . Thus  $d_k([e_\alpha^k])$  is the image the generator of  $H_{k-1}(S^{k-1})$  under the gluing map  $q_{k-1} \circ \partial \varphi_\alpha^k$ .

(4) Conclude the matrix coefficient of  $d_k$  from  $e_\alpha^k$  to  $e_\beta^{k-1}$  is  $\deg(f_{\alpha\beta})$ .

**Exercise 5.15.** For  $k = 1$ , notice that  $\phi_\gamma^1 : D^1 = I \rightarrow X^1$  is a path and  $\phi_\gamma^1(t) \in X^0$  for  $t = 0, 1$ . Show that<sup>2</sup>

$$d_1([e_\alpha^1]) = [\phi_\gamma^1(1)] - [\phi_\gamma^1(0)].$$

Hint: This is essentially a manipulation for the comparison of reduced  $\tilde{H}_0$  and non-reduced  $H_0$ , compare to Exercise 2.31 and Remark 4.29.

**Exercise 5.16.** Write down the cellular chain  $C_*(X)$  for  $X = S^n, T^2, \Sigma$  surface of genus  $g, \mathbb{C}P^n$ , and compute their homology (surely computing the cellular homology, but we know the result is isomorphic to the singular homology). Then compare your computation with Example 4.32. Notice, you shall give CW decomposition first!

**Exercise 5.17.** Suppose  $X$  has a  $\Delta$ -complex structure, show that for  $k \geq 1$  that

$$d([e_\alpha^k]) = \sum_{i=0}^k (-1)^i [\partial_i e_\alpha^k],$$

where  $\partial_i e_\alpha^k$  means the cell given by the map  $\varphi_\alpha^k \circ \delta_k^i$ . I.e the cellular boundary of  $e_\alpha^k$  is the usual simplicial boundary of the singular simplex  $\varphi_\alpha^k$ .

In this case, we also call the  $H_*^{CW}(X)$  the simplicial homology of  $X$ .

*Remark 5.18.* In this section, we mainly consider cellular homology as a computational tool. However, we may also develop an entire homology theory for cellular homology, for example, functoriality and long exact sequences. We left the discussion to interested readers.

### 5.3. Some fun applications.

**Exercise 5.19.** If  $X$  is a finite-type CW complex, show that  $H_q(X)$  is a finitely generated abelian group for all  $q$ . In particular, one can define Betti numbers for finite CW complexes.

**Exercise 5.20.** Recall Definition 3.21. For a finite-type CW complex  $X$ , denote  $c_q(X)$  as the rank of  $C_q(X)$ , i.e. the number of  $q$ -cells. Show that if  $X$  is finite, then

$$\chi(X) = \sum_q (-1)^q c_q(X).$$

This is the fancy version of  $2 = \chi(S^2) = V - E + F$ .

<sup>2</sup>In principle, we can state the  $d_1$  formula in terms degree as well. But then we need to be careful with the notion of degree for  $S^0$ , which is a little fretful.

## 6. KÜNNETH FORMULA

In this section, we discuss the product space and its homology.

**6.1. Alexander-Whitney map and Eilenberg-Zilber theorem.** To study  $S_*(X \times Y)$ , we are trying to compare it with  $S_*(X) \otimes S_*(Y)$ . Therefore, it means that for a singular simplex  $\sigma : \Delta_n \rightarrow X \times Y$ , we are trying to find some simplexes  $\Delta_p \rightarrow X$  or  $\Delta_{n-p} \rightarrow Y$ . We're gonna give those construction.

**Construction:** For  $n \geq 0$  and  $0 \leq p \leq n$ , we define two maps

$$[v_0, \dots, v_p] : \Delta^p \rightarrow \Delta^n, \quad [v_p, \dots, v_n] : \Delta^{n-p} \rightarrow \Delta^n,$$

i.e., the first  $p$ -dimension simplex and the last  $q = n - p$ -dimension simplex.

**Definition 6.1.** For singular simplex<sup>3</sup>  $\sigma : \Delta^n \rightarrow Z$  and  $0 \leq p \leq n$ , we define

$$p\sigma : \Delta^p \xrightarrow{[v_0, \dots, v_p]} \Delta^n \xrightarrow{\sigma} Z, \quad \sigma_{n-p} : \Delta^{n-p} \xrightarrow{[v_p, \dots, v_n]} \Delta^n \xrightarrow{\sigma} Z.$$

**Definition 6.2.** For  $X \times Y \xrightarrow{p_1} X$  and  $X \times Y \xrightarrow{p_2} Y$ , we define the Alexander-Whitney map

$$AW_n : S_n(X \times Y) \rightarrow (S_*(X) \otimes S_*(Y))_n, \quad \sigma \mapsto \sum_{p=0}^n p_{1\#}(p\sigma) \otimes p_{2\#}(\sigma_{n-p}).$$

*Supplement material 6.3.* The essence of the construction is the following question: For the diagonal map  $\delta : \Delta_p \rightarrow \Delta_p \times \Delta_p$ , it is clear that  $\Delta_p$  has a CW decomposition such that  $q$ -cells are  $q$ -faces of  $\Delta_p$ , and  $\Delta_p \times \Delta_p$  has a product CW decomposition (Exercise 6.13). But in this case,  $\delta$  does not map  $q$ -skeleton to  $q$ -skeleton. The work of Alexander-Whitney gives a map  $\delta' : \Delta_p \rightarrow \Delta_p \times \Delta_p$  such that  $\delta' \simeq \delta$  and  $\delta'$  respects the CW decomposition. And the above construction is deeply related to the construction of  $\delta'$ .

**Exercise 6.4.** Prove the following face identities

$$\begin{aligned} d_i(p\sigma) &= p_{-1}(d_i\sigma), & 0 \leq i \leq p-1, \\ d_j(\sigma_{n-p}) &= (d_{p+j}\sigma)_{n-p-1}, & 1 \leq j \leq n-p, \\ d_p(p\sigma) &= d_0(\sigma_{n-p}). \end{aligned}$$

Then show that the Alexander-Whitney map  $AW : S_*(X \times Y) \rightarrow S_*(X) \otimes S_*(Y)$  is a chain map, and moreover, if  $f : X \rightarrow X'$  and  $g : Y \rightarrow Y'$ , then  $f_{\#} \otimes g_{\#} \circ AW = AW \circ (f \times g)_{\#}$ .

**Theorem 6.5** (Eilenberg-Zilber). *The Alexander-Whitney map  $AW : S_*(X \times Y) \rightarrow S_*(X) \otimes S_*(Y)$  is a chain homotopy equivalence, i.e. there exists  $EM : S_*(X) \otimes S_*(Y) \rightarrow S_*(X \times Y)$  such that  $AW \circ EM \simeq \text{id}$  and  $EM \circ AW \simeq \text{id}$ .*

*Moreover, we can take  $EM$  such that for  $f : X \rightarrow X'$  and  $g : Y \rightarrow Y'$ , then  $(f \times g)_{\#} \circ EM = EM \circ f_{\#} \otimes g_{\#}$ .*

*Proof.* The proof needs some more fancy algebra, or careful geometry. We omit it in our course. We refer to [Die08, Section 9.7] and exercises therein.  $\square$

*Supplement material 6.6.* Both  $EM$  and the homotopies here are unique *upto (higher) homotopies*.

To the reader's convenience, we write down one very common formula of  $EM$ : We say a permutation  $\lambda$  of  $1, \dots, p+q$  is a  $(p, q)$ -shuffle if  $\lambda(1) < \dots < \lambda(p)$ , and  $\lambda(p+1) < \dots < \lambda(p+q)$ . For a  $(p, q)$ -shuffle, one can define a linear map  $[\lambda] : \Delta^{p+q} = [v_i]_i \rightarrow \Delta^p \times \Delta^q = [s_a]_a \times [t_b]_b$ . Let  $a_i = |\{1 \leq k \leq p \mid \sigma(k) \leq i\}|$ ,  $b_i = |\{1 \leq l \leq q \mid \sigma(p+l) \leq i\}|$ , then  $[\lambda]$  is the linear extension the assignment

$$[\lambda](v_i) = (s_{a_i}, t_{b_i}).$$

Then we set

$$EM : S_*(X) \otimes S_*(Y) \rightarrow S_*(X \times Y), \quad \sigma \otimes \tau \mapsto \sum_u \text{sign}(\lambda)(\sigma \times \tau) \circ [\lambda],$$

<sup>3</sup>Here, we use  $Z$  to indicate that the construction does not only work for  $X \times Y$ , or  $X, Y$ .

where  $\lambda$  runs for all  $(p, q)$ -shuffles.

If you know the wedge of differential form, you are probably familiar with this formula. Also, the decomposition shown in the proof of the homotopy invariance is a special case here by taking  $q = 1$ .

*Supplement material 6.7.* Another interesting point is that both *AW* and *EM* works in some way for simplicial sets. Then we can actually show that the Dold-Kan correspondence (Supplement material 2.13) is a monoidal functor in a certain sense.

*Remark 6.8.* Here, we should be careful on the coefficient. If we consider  $S_*(-; R)$  for a commutative ring  $R$  and treat  $S_*(-; R)$  as chain complex over  $R$ , then we also have  $S_*(X; R) \otimes_R S_*(Y; R) \simeq S_*(X \times Y; R)$  since  $S_*(X)$  is a free chain complex.

**Theorem 6.9** (Künneth formula). *For  $X, Y$  be two space, we have a short exact sequence*

$$0 \rightarrow \bigoplus_{p+q=r} H_p(X) \otimes H_q(Y) \rightarrow H_r(X \times Y) \rightarrow \bigoplus_{p+q=r-1} \text{Tor}_1^{\mathbb{Z}}(H_p(X), H_q(Y)) \rightarrow 0.$$

*Proof.* Under Theorem 6.5, we have chain homotopy equivalence  $S_*(X) \otimes S_*(Y) \simeq S_*(X \times Y)$ .

Notice that  $S_*(X)$  and  $S_*(Y)$  are free chain complex, so  $S_*(X) \otimes S_*(Y)$  computes the derived tensor product. Then we can use the algebraic Künneth formula Theorem A.19.  $\square$

*Remark 6.10.* Here, we refer to Remark 2.29 for the black-box of  $\text{Tor}_1^{\mathbb{Z}}$ . The theorem has a generalization for all commutative rings  $R$ , but the statement is more subtle: the above still argument work but the subtlety is when you computing a derived tensor product over  $R$ , you need a spectral sequence for higher  $\text{Tor}_k^R$ . The simpleness of  $\mathbb{Z}$  is it is a PID, so only  $\text{Tor}_1^{\mathbb{Z}}$  non-zero; and for a field  $\mathbb{F}$ ,  $\text{Tor}_k^{\mathbb{F}} = 0$  for all  $k$ .

Therefore, for a field  $\mathbb{F}$ , we actually have

$$\bigoplus_{p+q=r} H_p(X; \mathbb{F}) \otimes_{\mathbb{F}} H_q(Y; \mathbb{F}) \cong H_r(X \times Y; \mathbb{F}).$$

**Exercise 6.11.** Suppose for  $X, Y$  we have  $H_p(X), H_q(Y)$  are finitely generated abelian groups for all  $p, q$ , then  $H_r(X \times Y)$  is finitely generated abelian group for all  $r$ .

In particular, we have  $b_r(X \times Y) = \sum_{p+q=r} b_p(X) b_q(Y)$ .

**Exercise 6.12.** Show that  $H_q(T^n) \cong \mathbb{Z}^d$  where  $d = \binom{n}{q}$ .

**6.2. Product of CW complex.** Though the general product theorem is harder to prove. The case for CW complex is more intuitive. Here, we give the explanation.

**Exercise 6.13.** Let  $X, Y$  be two finite-type CW complexes with the set of cells  $\{e_i^p\}$  and  $\{e_j^q\}$ . We define  $(X \times Y)^r = \cup_{p+q=r} X^p \times Y^q$  with the cells set  $\{e_i^p \times e_j^q\}$ , and characteristic maps for  $e_i^p \times e_j^q$  by  $\varphi_i^p \times \varphi_j^q : D^{p+q} \cong D^p \times D^q \rightarrow X \times Y$ . Show<sup>4</sup> that this is indeed a finite-type CW decomposition of  $X \times Y$ . Try to see the CW decomposition of  $T^2$  we gave in Example 4.32 is the one that comes from the product of  $S^1$  (with the evident CW decomposition).

**Exercise 6.14.** \*\* Show the following differential formula

$$d_{p+q}(e_i^p \times e_j^q) = (d_p e_i^p) \times e_j^q + (-1)^p e_i^p \times (d_q e_j^q).$$

Hint: The technique is similar to Exercise 5.14. But the mystery sign  $(-1)^p$  comes from choices of orientation of spheres (i.e., choice of the generator of  $H_n(S^n)$ ).

**Exercise 6.15.** For  $X, Y$  be two finite-type CW complexes, and equip  $X \times Y$  the product CW decomposition of Exercise 6.13. Show that the following map induces isomorphism of chain complex

$$C_*(X \times Y) \rightarrow C_*(X) \otimes C_*(Y), \quad e_i^p \times e_j^q \mapsto e_i^p \otimes e_j^q.$$

<sup>4</sup>The product decomposition still gives a CW decomposition if one of  $X$  or  $Y$  is finite-type or both  $X$  and  $Y$  have countably many  $q$ -cells for all  $q$ , but it is more tricky.

Consequently, we have

**Theorem 6.16.** *For  $X, Y$  be two finite-type CW complexes, we have a short exact sequence*

$$0 \rightarrow \bigoplus_{p+q=r} H_p^{CW}(X) \otimes H_q^{CW}(Y) \rightarrow H_r^{CW}(X \times Y) \rightarrow \bigoplus_{p+q=r-1} \text{Tor}_1^{\mathbb{Z}}(H_p^{CW}(X), H_q^{CW}(Y)) \rightarrow 0.$$

*Then by Theorem 5.11, you may obtain a different proof of Theorem 6.9 for finite-type CW complexes.*

Notice that the discussion of Remark 6.10 for the coefficient also applies.

*Proof.* Under Exercise 6.15, we can use the algebraic Künneth formula Theorem A.19.  $\square$

## 7. COHOMOLOGY AND CUP PRODUCT

Now, we study cohomology theory. Superficially, cohomology gains no knowledge compared to homology as groups. However, we will see later that we can equip a ring structure on cohomology, which makes it essentially better than homology!

For example, we shall know that  $T^2$  and  $S^2 \vee S^1 \vee S^1$  are not homotopy equivalent (now you should know that both of them have the homology groups  $H_0 = \mathbb{Z}, H_1 = \mathbb{Z}^2, H_2 = \mathbb{Z}$ , and you will see that the cohomology groups are the same as well) by showing their ring structure are different.

**7.1. Singular cohomology.** We define singular cohomology and explain the cohomological versions of many theorems we have previously proved. **Notice:** this section could also be a review sheet for what we have learned so far (parallel to the homology version.)

**Recall:** For two abelian groups  $A, M$ , we have the dual pairing

$$\langle -, - \rangle : \text{Hom}_{\mathbb{Z}}(A, M) \times A \rightarrow M, \langle \alpha, a \rangle := \alpha(a).$$

For a homomorphism  $f : A \rightarrow B$  between abelian groups and an abelian group  $M$ , we can define its dual  $f^\vee : \text{Hom}_{\mathbb{Z}}(B, M) \rightarrow \text{Hom}_{\mathbb{Z}}(A, M)$  as following, for  $\beta \in \text{Hom}_{\mathbb{Z}}(B, M)$

$$f^\vee(\beta)(a) = \beta(f(a)),$$

equivalently, we can write it using the pairing

$$\langle f^\vee(\beta), a \rangle = \langle \beta, f(a) \rangle$$

for all  $(\beta, a) \in \text{Hom}_{\mathbb{Z}}(B, M) \times A$ .

**Exercise 7.1.** Here, we put the algebraic result as an exercise. Consider the short exact sequence of abelian groups

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,$$

and we may take its dual exact sequence

$$\text{Hom}_{\mathbb{Z}}(A, M) \leftarrow \text{Hom}_{\mathbb{Z}}(B, M) \leftarrow \text{Hom}_{\mathbb{Z}}(C, M) \leftarrow 0,$$

which is generally not extended to a exact sequence at  $\text{Hom}_{\mathbb{Z}}(A, M)$ .

Show that if  $C$  is free, then the dual exact sequence can be extended to SES

$$0 \leftarrow \text{Hom}_{\mathbb{Z}}(A, M) \leftarrow \text{Hom}_{\mathbb{Z}}(B, M) \leftarrow \text{Hom}_{\mathbb{Z}}(C, M) \leftarrow 0.$$

You may prove that if  $C$  is free, then the SES  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is isomorphic to  $0 \rightarrow A \rightarrow A \oplus C \rightarrow C \rightarrow 0$  (i.e. the SES split), and then can show that split SES are mapped to SES after taking dual  $\text{Hom}_{\mathbb{Z}}(-, M)$ . From the argument, you also learn that this is a result about the dual of split SES, and  $C$  free is just a sufficient condition for splitting the SES.

**Definition 7.2.** For a space  $X$  and an abelian group  $M$ , we set

$$S^q(X; M) := \text{Hom}_{\mathbb{Z}}(S_q(X), M), \quad \delta^q := \partial_{q+1}^\vee : S^q(X; M) \rightarrow S^{q+1}(X; M).$$

Then we call the cochain complex  $S^*(X; M) = (S^p(X; M), \delta^p)$  the singular cochain of  $X$ , and we set abelian groups

$$B^q(X; M) := \text{im}(\delta_{q-1}) \subset \ker(\delta_q) =: Z^q(X; M)$$

and define the  $q$ -th **singular cohomology** group as the quotient

$$H^q(X; M) := Z^q(X; M) / B^q(X; M).$$

We call cochains in  $Z^q(X; M)$  cocycles and cochains in  $B^q(X; M)$  coboundaries.

If  $M = \mathbb{Z}$ , we omit  $M$  from the notation: For example,  $S^*(X)$ ,  $H^q(X)$  and so on.

*Remark 7.3.* By definition, a cochain  $c \in S^q(X; M)$  is a  $\mathbb{Z}$ -linear function<sup>5</sup>

$$c : S_q(X) \rightarrow M.$$

And  $c$  is a cocycle means that the restriction of  $c$  on the group of boundary chains  $B_q(X)$  is 0.

Now, let us give theorems, and to give you some key point for their proof. We only state them for  $\mathbb{Z}$  for simplicity.

**Theorem 7.4.** *For a map  $f : X \rightarrow Y$ , we have induced cochain map  $S^*(f) = f^\#$  and its cohomology  $H^q(f) = f^q$*

$$f^\# : S^*(Y) \rightarrow S^*(X), \quad f^q : H^q(Y) \rightarrow H^q(X),$$

*which are compatible with composition.*

*When  $f \simeq g$ , we have  $f^\# \simeq g^\#$  (as cochain maps), and then  $f^q = g^q$ .*

*Consequently, cohomology are homotopy invariants.*

*Proof.* The cochain map  $f^\# = S^*(f)$  is simply  $(S_*(f))^\vee$ , and then  $H^q(f) = f^q$  are cohomology of  $S^*(f)$ . Then the homotopy invariance also follows from the result for chains (it is clear a dual of a chain homotopy is also a cochain homotopy).  $\square$

**Remark 7.5. Important!** The main significant difference between (singular) cohomology and homology is that: the direction of  $S^*(f) = f^\#$  and  $H^q(f) = f^q$ , which is opposite to  $f_\#$  and  $f_q$ . It means that cohomologies are contravariant functors while homologies are covariant functors.

The difference is so tiny, but so important in many applications. Especially, the product structure on cohomology follows from this reason! (But also, sometimes it is a bad thing to have a contravariant functor, you may learn the defect for cohomology in your life sometime.)

In principle, no need to write a cochain small chain theorem. We will only write their corollary, i.e. the Mayer-Vietoris sequence and the excision.

**Theorem 7.6.** *We have the Mayer-Vietoris long exact sequences: If  $\{X_1, X_2\}$  is a Mayer-Vietoris duo, then there exists a long exact sequence*

$$\cdots \rightarrow H_q(X_1 \cup X_2) \xrightarrow{a_q} H_q(X_1) \oplus H_q(X_2) \xrightarrow{s_q} H_q(X_1 \cap X_2) \xrightarrow{\delta} H_{q+1}(X_1 \cup X_2) \rightarrow \cdots .$$

*And the sequence is natural with respect to maps between Mayer-Vietoris duos.*

*Proof.* Compare to the proof of Theorem 4.6, we only need to show that

$$0 \rightarrow \text{Hom}_{\mathbb{Z}}(S_*(X_1) + S_*(X_2), \mathbb{Z}) \xrightarrow{a^\#} S_*(X_1) \oplus S_*(X_2) \xrightarrow{s^\#} S^*(X_1 \cap X_2) \rightarrow 0$$

is  $\text{Hom}_{\mathbb{Z}}(S_*(X_1) + S_*(X_2), \mathbb{Z}) \simeq S^*(X_1 \cup X_2)$  and the sequence is exact.

The former follows from the fact that the small chain theorem gives a chain homotopy, so after taking dual  $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ , we have a cochain homotopy. For the latter, we use the following fact that  $S_*(X_1) \oplus S_*(X_2)$  is a degree-wise free chain complex, and then apply Exercise 7.1!  $\square$

**Corollary 7.7.** *Let  $f : X \rightarrow Y$  be a map, then there exists a long exact sequence*

$$\cdots \rightarrow H^q(C(f)) \xrightarrow{e^q} H^q(Y) \xrightarrow{f^q} H^q(X) \xrightarrow{\delta} H^{q+1}(C(f)) \rightarrow \cdots ,$$

*where  $e : Y \rightarrow C(f)$  is the inclusion map.*

Next, we consider the cellular cohomology

**Definition 7.8.** For a CW complex  $X$ , we define its cellular cochain complex by

$$C^*(X; M) = (\text{Hom}_{\mathbb{Z}}(C_q(X), M), d_q^\vee).$$

Its cohomology groups are called the cellular cohomology of  $X$ , denoted by  $H_{CW}^q(X, M)$ .

**Theorem 7.9.** *We have  $H_{CW}^q(X) \simeq H^q(X)$  for a CW complex  $X$ .*

<sup>5</sup>You can imagine you put some color on different piece of you space, the intuition is an early motivation for cohomology theory.

Notice that we do not have  $C^*(X; M) \simeq S^*(X; M)$  here, but you can run the same argument for Theorem 5.11 to prove the theorem.

**Exercise 7.10.** On the computational results, we list them all here as exercise:

(1) For a contractible space  $X$ , we have

$$H^q(X) \cong 0, \forall q \geq 1, \quad H^0(X) \cong \mathbb{Z}.$$

(2) We have  $H^q(\bigsqcup_i X_i) \cong \prod_i H^q(X_i)$ ;  $H^q(X) \cong \tilde{H}^q(X), \forall q \geq 1, \quad H^0(X) \cong \tilde{H}^0(X) \oplus \mathbb{Z}$ ;  
and  $\tilde{H}^*(\bigvee_i^d X_i) = \prod_i \tilde{H}^*(X_i)$ .<sup>6</sup>

(3) The space  $X$  is path connected if and only if  $H^0(X) \cong \mathbb{Z}$ .

(4) For sphere,  $n \geq 1$ ,

$$\tilde{H}^q(S^n) \cong 0, \forall q \neq n, \quad H^n(S^n) \cong \mathbb{Z}$$

(5) For the genus  $g$  surface  $\Sigma$ , we have  $H^2(\Sigma) \cong \mathbb{Z}, \quad H^1(\Sigma) \cong \mathbb{Z}^{2g}, H^0(\Sigma) \cong \mathbb{Z}$ .

(6)  $H^q(\mathbb{C}P^n) = \mathbb{Z}$  for  $q = 0, 2, \dots, 2n$  and trivial otherwise.

Hint: You may repeat our argument for homology. But a shortcut is using the universal coefficient theorem below.

At the end, we present another universal coefficient theorem, which enables you to compute the cohomology *group* from homology groups. We start with a warm-up exercise.

**Exercise 7.11.** The chain level dual pairing  $S^p(X; M) \otimes S_p(X) \rightarrow M, \langle c, \sigma \rangle = c(\sigma)$  descend to  $\kappa : H^p(X; M) \otimes H_p(X) \rightarrow M$ . (Noticed that the (co)homology level pairing is not perfect, so we do NOT have  $H^p(X) \cong H_p(X)$  in general.)

Consequently, we have a homomorphism between abelian groups

$$\text{ev}_H : H^q(X; M) \rightarrow \text{Hom}_{\mathbb{Z}}(H_q(X), M).$$

**Theorem 7.12** (Universal coefficient theorem for cohomology). *Let  $M$  be an abelian group and  $X$  be a space, for each  $q$ , we have the following short exact sequence*

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{q-1}(X), M) \rightarrow H^q(X; M) \xrightarrow{\text{ev}_H} \text{Hom}_{\mathbb{Z}}(H_q(X), M) \rightarrow 0.$$

*Proof.* Noticed that  $S_*(X)$  is a free chain complex, then you can apply the algebraic universal coefficient theorem for cohomology, i.e. Theorem A.18 to  $S^*(X; M)$ .  $\square$

*Remark 7.13.* Similar to Remark 2.29, if you don't know  $\text{Ext}_{\mathbb{Z}}^1$ , you may use it directly based on the following information: 1)  $\text{Ext}_{\mathbb{Z}}^1$  commutes with finite direct sum on both variables. 2)  $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/\text{gcd}(m, n)$ . 3)  $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n, M) = 0$  if  $M$  is projective, for example  $\mathbb{Z}$ .

**Exercise 7.14.** You can also define  $b^q(X)$  as  $\text{rank} H^q(X)$  when  $H^q(X)$  is finitely generated.

Show that  $H^q(X)$  is finitely generated  $\forall q$  if and only if  $H_q(X)$  is finitely generated  $\forall q$ , and  $b^q(X) = b_q(X) \forall q$ . So, it doesn't matter whether you are defining the Betti number using homology or cohomology.

**7.2. Products on cohomology.** Now, we introduce the ring structure on cohomology.

**Definition 7.15.** Let  $R$  to be a commutative ring, and we denote  $m : R \otimes R \rightarrow R, (x, y) \mapsto xy$  the ring multiplication.

Let  $X, Y$  be a spaces, we define the chain level cross-product as the composition

$$\begin{aligned} \times : S^*(X; R) \otimes S^*(Y; R) &= \text{Hom}_{\mathbb{Z}}(S_*(X), R) \otimes \text{Hom}_{\mathbb{Z}}(S_*(Y), R) \\ &\xrightarrow{\otimes} \text{Hom}_{\mathbb{Z}}(S_*(X) \otimes S_*(Y), R \otimes R) \\ &\xrightarrow{m} \text{Hom}_{\mathbb{Z}}(S_*(X) \otimes S_*(Y), R) \\ &\xrightarrow{AW^\vee} \text{Hom}_{\mathbb{Z}}(S_*(X \times Y), R) = S^*(X \times Y; R), \end{aligned}$$

<sup>6</sup>Be careful here, for homology, we get direct sum, but for cohomology we get direct product. Surely they are the same if there are finitely many factors, but different in general.

where  $AW^\vee$  is the dual of the Alexander-Whitney map.

If  $X = Y$ , we define the chain level cup-product as

$$\smile : S^*(X; R) \otimes S^*(X; R) \xrightarrow{\times} S^*(X \times X; R) \xrightarrow{\Delta^\#} S^*(X; R),$$

where  $\Delta : X \rightarrow X \times X, x \mapsto (x, x)$  is the diagonal map.

*Remark 7.16.* Now, since we need to use multiplication to define the product, we need to take the coefficient not just an abelian group, but also a commutative ring. It is clear that both the cross product and the cup product are  $R$ -bilinear. In below, we use  $R = \mathbb{Z}$  to simplify notation.

**Exercise 7.17.** Let us keep track of the process, and show the following formula for cup product. Let  $c \in S^p(X)$ ,  $d \in S^q(X)$  and  $\sigma \in S_{p+q}(X)$ , we have (recall Definition 6.1)

$$(c \smile d)(\sigma) = c(p\sigma) \cdot d(\sigma_q).$$

Hint: Use the explicit formula for  $AW$  and notice that  $p_i\Delta = \text{id}_X$  for  $i = 1, 2$ .

**Notice** If you don't want to know what the Alexander-Whitney map is, the formula could be regarded as the definition of cup product.

**Exercise 7.18.** Show that we can compute the cross product using cup product, for  $c \in S^p(X)$ ,  $d \in S^q(Y)$ , we have

$$c \times d = p_X^\# c \smile p_Y^\# d.$$

**Proposition 7.19.** *The chain level cross/cup product satisfies the Leibnitz rule:  $c \in S^p(X)$ ,  $d \in S^q(Y)$  ( $X = Y$  when discussing  $\smile$ ), then we have*

$$\delta(c \times d) = \delta c \times d + (-1)^p c \times \delta d, \quad \delta(c \smile d) = \delta c \smile d + (-1)^p c \smile \delta d.$$

*Proof.* For the cross product, the result essentially follows from the fact that  $AW$  is a chain map, and the Leibnitz rule for graded tensor product. We present the details here.

We denote the composition of  $m$  and  $\otimes$  as  $\otimes_m$ , then

$$(c \times d)(\psi) = (c \otimes_m d)(AW(\psi)), \quad \psi \in S_{p+q}(X \times Y).$$

Since  $AW$  is a chain map, for  $\sigma \in S_{p+q+1}(X \times Y)$  we have

$$(\delta(c \times d))(\psi) = (c \times d)(\partial\psi) = (c \otimes_m d)(AW(\partial\psi)) = (c \otimes_m d)(\partial AW(\psi)) = (\delta(c \otimes_m d))(AW(\psi)).$$

On the tensor product cochain complex,

$$\delta(c \otimes_m d) = \delta c \otimes_m d + (-1)^p c \otimes_m \delta d.$$

Therefore

$$\begin{aligned} (\delta(c \times d))(\sigma) &= (\delta c \otimes_m d + (-1)^p c \otimes_m \delta d)(AW(\sigma)) \\ &= (\delta c \times d + (-1)^p c \times \delta d)(\sigma). \end{aligned}$$

Hence

$$\delta(c \times d) = \delta c \times d + (-1)^p c \times \delta d.$$

Next, for the cup product, we have

$$\delta(c \smile d) = \delta\Delta^\#(c \times d) = \Delta^\#\delta(c \times d) = \Delta^\#(\delta c \times d + (-1)^p c \times \delta d) = \delta c \smile d + (-1)^p c \smile \delta d. \quad \square$$

Therefore, both the cross product and cup product descend to cohomology.

**Definition 7.20.** Let  $X$  be a space, we define the cohomology level cross-product and cup-product

$$\begin{aligned} \times : H^*(X) \otimes H^*(Y) &\rightarrow H^*(X \times Y), \quad [c] \times [d] \mapsto [c \times d] \\ \smile : H^*(X) \otimes H^*(X) &\rightarrow H^*(X), \quad [c] \smile [d] \mapsto [c \smile d]. \end{aligned}$$

**Exercise 7.21.** Prove that the cohomology level cross-product and cup-product are well-defined using the Leibniz rule. Moreover, we have  $\Delta^*([c] \times [d]) = [c] \smile [d]$  and  $[c] \times [d] = p_X^*[c] \smile p_Y^*[d]$ .

Now, we give the first example.

**Example 7.22.** On  $H^*(S^n)$ , let  $x$  be a generator of  $H^n(S^n)$ , then by degree reason

$$x \smile x = 0.$$

Then we have  $H^*(S^n) \cong \mathbb{Z}[x]/(x^2)$  where  $|x| = n$ .

We can use the cross product to give a Künneth formula for cohomology.

**Theorem 7.23.** For  $X, Y$  be two spaces, and if  $H_q(Y)$  are finitely generated for all  $q$  (for example  $Y$  finite-type CW complex), we have a short exact sequence

$$0 \rightarrow \bigoplus_{p+q=r} H^p(X) \otimes H^q(Y) \xrightarrow{\smile} H^r(X \times Y) \rightarrow \bigoplus_{p+q=r+1} \text{Tor}_1^{\mathbb{Z}}(H_p(X), H_q(Y)) \rightarrow 0;$$

Or over a field  $\mathbb{F}$  (but still need finiteness), we have

$$\bigoplus_{p+q=r} H^p(X; \mathbb{F}) \otimes_{\mathbb{F}} H^q(Y; \mathbb{F}) \cong H^r(X \times Y; \mathbb{F}).$$

*Remark 7.24.* It is a little weird to require a finiteness condition for the cohomological Künneth formula. This is because the cohomology theory is a dual! We often need finiteness to make dual behavior not too bad anyway.

**Example 7.25.** Here, we use Theorem 7.23 to compute the cup product on  $H^*(T^2)$ . The only non-trivial product is  $H^1(T^2) \otimes H^1(T^2) \rightarrow H^2(T^2)$  (except the degree reason, we still have  $H^0(T^2) \otimes H^*(T^2) \rightarrow H^*(T^2)$ , but you will see that this is clear by the fact that the cup product is unital).

In fact, by the Künneth formula theorem, we have that

$$H^1(S_1^1) \otimes H^1(S_2^1) \xrightarrow{\cong} H^2(T^2),$$

where you see that the cross product gives the homomorphism. Then we have that the generator  $b$  of  $H^2(T^2)$  is given by  $b = a_1 \times a_2$  for  $H^*(S_i^1) = \mathbb{Z}a_i$ .

But, by geometric construction, two generators of  $H^1(T^2)$  are  $u_i = p_i^*a_i$ . Then  $u_1 \smile u_2 = a_1 \times a_2 = b$  by Exercise 7.18. In summary, we have  $H^*(T^2) = \mathbb{Z}[x, y]/(x^2, y^2)$  pour  $|x| = |y| = 1$  as graded rings.

In this way, we see inductively that the isomorphism of graded commutative algebras  $H^*(T^n) \cong \mathbb{Z}[x_1, \dots, x_n]/(x_i^2 \mid i = 1, \dots, n)$  the exterior algebra generated by  $n$  many degree 1 elements.

At the end, we introduce the cap product, which is useful for the study of manifolds. Now, I believe you can finish the construction as an exercise.

**Exercise 7.26.** We define the cap product

$$\frown: S^p(X) \otimes S_n(X) \rightarrow S_{n-p}(X),$$

on singular simplex  $\sigma$  by

$$c \frown \sigma = c_{(p)\sigma} \cdot \sigma_{n-p},$$

and we extend it linearly to all singular chains  $\sigma$ .

Show the following

(1) The chain level boundary formula: for  $c \in S^p$ , we have

$$\partial(c \frown \sigma) = (-1)^p(\delta c \frown \sigma - c \frown \partial\sigma).$$

Therefore,  $\frown$  descends to a cap product on (co)homology.

$$\frown: H^q(X) \otimes H_n(X) \rightarrow H_{n-q}(X).$$

(2) On the chain level, we have the formula

$$\langle c \smile d, \sigma \rangle = \langle c, d \frown \sigma \rangle.$$

Then show the same formula descends to (co)homology level.

**Warning:** For the terms (3), (4) below, I suggest you do it after reading the next section. We put it here to keep the structure of the notes compact.

(3) Let  $e \in S^0(X)$  be the chain level cup product unit, i.e. the singular cochain constantly equals 1 on  $S_0(X)$ . Then we have  $e \smile \sigma = \sigma$  for  $\sigma \in S_*(X)$ .

(4) Let  $f : X \rightarrow Y$  be a continuous map, we have the projection formula for the cap product:

$$f_{\#}(f^{\#}c \frown \sigma) = c \frown f_{\#}\sigma.$$

Everything works  $R$ -linearly.

**7.3. Cup product forms a ring.** Now, let us explain some properties of the cup product. I would suggest you skip the proof in your first reading.

**Theorem 7.27.** *Let  $X$  be a space, we have that the chain level cup-product is unital, associative, in particular,  $(S^*(X), \smile)$  is a unital ring. If  $f : X \rightarrow Y$  is a map, then  $f^{\#}$  is a ring homomorphism.*

*It descends to cohomology, and then also has that  $(H^*(X), \smile)$  is a unital ring, and  $f^* : H^*(Y) \rightarrow H^*(X)$  is a ring homomorphism.*

*Proof.* We only prove the chain level result. All those laws descend to cohomology.

Recall the formula for cup product (Exercise 7.17)

$$(c \smile d)(\sigma) = c({}_p\sigma) d({}_q\sigma), \quad c \in S^p(X), \quad d \in S^q(X), \quad \sigma \in S_{p+q}(X).$$

We first show associativity. Let  $a \in S^p(X)$ ,  $b \in S^q(X)$  and  $c \in S^r(X)$ , and let  $\sigma \in S_{p+q+r}(X)$ . Then

$$\begin{aligned} ((a \smile b) \smile c)(\sigma) &= (a \smile b)({}_{p+q}\sigma) c({}_r\sigma) \\ &= a({}_p({}_{p+q}\sigma)) b({}_q({}_{p+q}\sigma)) c({}_r\sigma). \end{aligned}$$

On the other hand,

$$\begin{aligned} (a \smile (b \smile c))(\sigma) &= a({}_p\sigma) (b \smile c)({}_{q+r}(\sigma)) \\ &= a({}_p\sigma) b({}_q({}_{q+r}(\sigma))) c({}_{r}({}_{q+r}(\sigma))). \end{aligned}$$

Now the relevant faces are the same simplex:

$${}_p({}_{p+q}\sigma) = {}_p\sigma, \quad ({}_{p+q})_q = {}_q({}_{q+r}(\sigma)), \quad ({}_{q+r})_r = {}_r\sigma.$$

Hence

$$((a \smile b) \smile c)(\sigma) = (a \smile (b \smile c))(\sigma),$$

so  $\smile$  is associative.

Next, let  $e \in S^0(X)$  be the constant 0-cochain sending every singular 0-simplex to  $1 \in \mathbb{Z}$ . Then for  $c \in S^p(X)$  and  $\sigma \in S_p(X)$ ,

$$(e \smile c)(\sigma) = e({}_0\sigma) c({}_p\sigma) = c(\sigma), \quad (c \smile e)(\sigma) = c({}_p\sigma) e({}_0\sigma) = c(\sigma).$$

Thus  $e$  is a two-sided unit, and  $S^*(X)$  is a unital graded ring.

If  $f : X \rightarrow Y$  is continuous, then for  $c \in S^p(Y)$ ,  $d \in S^q(Y)$  and  $\sigma \in S_{p+q}(X)$ ,

$$\begin{aligned} f^{\#}(c \smile d)(\sigma) &= (c \smile d)(f_{\#}\sigma) \\ &= c({}_p(f_{\#}\sigma)) d({}_q(f_{\#}\sigma)) \\ &= c(f_{\#}({}_p\sigma)) d(f_{\#}({}_q\sigma)) \\ &= (f^{\#}c)({}_p\sigma) (f^{\#}d)({}_q\sigma) \\ &= ((f^{\#}c) \smile (f^{\#}d))(\sigma). \end{aligned}$$

So  $f^{\#}$  is a ring homomorphism. □

*Remark 7.28.* Here, we mention that  $H^*(X)$  is not simply some abelian groups, and we should think it as  $H^*(X) = \bigoplus_q H^q(X)$ , a graded abelian group. The cup product respects the degree, so  $H^*(X)$  is a graded ring. And  $f^*$  preserves the degree, so  $f^*$  is a homomorphism between graded rings. Also,  $(S^*(X), \smile)$  is a differential graded algebra.

Notice that all of the operations are  $R$ -linear if  $R$ -coefficients show up.

**Exercise 7.29.** Similarly, show that the chain level cross product is also associative.

**Theorem 7.30.** *The cohomology ring  $(H^*(X), \smile)$  is graded commutative, i.e. for  $a \in H^p(X), b \in H^q(X)$ , we have*

$$a \smile b = (-1)^{pq} b \smile a.$$

*Proof.* To prove the theorem, we give a more precise result on the chain level.

For  $c \in S^p(X)$  and  $d \in S^q(X)$ , we define the cup-1 product

$$c \smile_1 d \in S^{p+q-1}(X)$$

is defined by

$$(c \smile_1 d)(\sigma) = \sum_{j=0}^{p-1} (-1)^{(p-j)(q+1)} d(\sigma| [v_0, \dots, v_j, v_{j+q}, \dots, v_{p+q-1}]) d(\sigma| [v_j, \dots, v_{j+q}]),$$

for every singular simplex  $\sigma : \Delta^{p+q-1} \rightarrow X$ . Here  $[\dots]$  means the sub-simplex spanned by the given vertices in  $\Delta^{p+q-1}$

The cup-1 product as the bilinear map

$$\smile_1 : S^p(X) \otimes S^q(X) \rightarrow S^{p+q-1}(X)$$

satisfying the relation (its proof is similar to many differential formulas we given before, but computation more involved, left it as an exercise)

$$\delta(c \smile_1 d) = \delta c \smile_1 d + (-1)^p c \smile_1 \delta d + (-1)^p c \smile d - (-1)^{pq+q} d \smile c.$$

Rearranging the identity gives

$$c \smile d - (-1)^{pq} d \smile c = \delta(c \smile_1 d) - \delta c \smile_1 d - (-1)^p c \smile_1 \delta d.$$

Now assume that  $c, d$  are cocycles. Then  $\delta c = \delta d = 0$ , so the above identity reduces to

$$c \smile d - (-1)^{pq} d \smile c = \delta(c \smile_1 d).$$

Thus  $c \smile d$  and  $(-1)^{pq} d \smile c$  differ by a coboundary, hence define the same cohomology class.  $\square$

**Exercise 7.31.** \* Prove the identity

$$\delta(c \smile_1 d) = \delta c \smile_1 d + (-1)^p c \smile_1 \delta d + (-1)^p c \smile d - (-1)^{pq+q} d \smile c.$$

This computation is similar to proving  $AW$  is a chain map. But a little long.

Notice that, in the proof of the equation, we need the coefficient ring to be commutative!

*Supplement material 7.32.* In many textbooks, the graded commutativity is proven by introducing a certain chain homotopy.

Here, we present the proof using the cup-1 product, which is conceptually hard (especially, I cannot explain what forces you to write down the formula), but somehow the computation is more straightforward.

The reason for using this approach is that we want to present that the graded commutativity actually comes from the fact

$$S^*(X) \text{ is a } \mathbb{E}_\infty \text{ algebra.}$$

In fact, we can define higher cup products  $\smile_i : S^*(X) \otimes S^*(X) \rightarrow S^*(X)[i]$  for all  $i \geq 0$  from the combinatorial structure of simplexes (where  $\smile_0 = \smile$ ), and they are organized in a way that failure of  $\smile_i$  product on chain level is measured by  $\smile_{i+1}$  products. Another interesting application is that the famous Steenrod square is computed by  $Sq^i([x]) = [x \smile_{q-i} x]$  for  $[x] \in H^q(X; \mathbb{F}_2)$ .

They further indicate fruitful structures on  $S^*(X)$ : ALL their composition and differential formulas (and their multi-entries versions) are organized in “the”  $\mathbb{E}_\infty$  operad. And then we say  $S^*(X)$  is an  $\mathbb{E}_\infty$  algebra [MS03].

I hope the proof motivates you to view the concept of an  $\mathbb{E}_\infty$ -algebra as a generalization of commutative algebra in the homotopy-coherent sense.

In practice, it is hard to convince many people why  $\mathbb{E}_\infty$  algebra is a meaningful notion. Here, I would like to slightly convince you by this proof.

*Supplement material 7.33.* You might have heard the de Rham theorem: For a smooth manifold  $M$ , we have

$$H_{dR}^q(M) \cong H^q(M; \mathbb{R}).$$

Here are some interesting things.

1) First, let me explain its proof. [Bre93, Chapter 3]

By some smooth approximation theorem, you can show the inclusion is a quasi-isomorphism

$$S_*^\infty(M) \rightarrow S_*(M),$$

where  $S_*^\infty(M)$  is the sub-chain generated by smooth simplexes  $\sigma : \Delta^p \rightarrow M$ .

Therefore, we have that  $S_\infty^*(M) = \text{Hom}_{\mathbb{R}}(S_*^\infty(M), \mathbb{R})$  is quasi-isomorphic to  $S^*(M)$ .

The point for  $S_*^\infty(M)$  is that we can integral

$$I : \Omega^p(M) \rightarrow S_\infty^p(M), \quad \omega \mapsto I(\omega) = [\sigma \mapsto \int_{\Delta^p} \sigma^* \omega].$$

The Stokes theorem tells you that:  $I$  is a chain map! And then descend to a map on cohomology, say  $H^q(I)$ .

You can easily prove that  $I$  is a quasi-isomorphism for open convex sets (with respect to a Riemannian metric) since convex sets are contractible.

Then you can take a covering of  $M$  by open convex sets, and then conclude using the Mayer-Vietoris sequences for both  $H_{dR}^*$  and  $H^*$  and the 5-lemma.

2) You can prove that  $H^*(I)$  is a ring homomorphism, which is helpful for you to compute the cup product. However, this is not completely obvious as  $I$  is NOT a ring homomorphism on the *chain level*. We have to work on the cohomology level: by the Fubini theorem, you can see that  $H^*(I)$  is compatible with the cross product on the cohomology level; and  $H^*(I)$  commutes with  $\Delta^*$  on both the de Rham side and the singular side. Then we conclude by Exercise 7.21.

**7.4. Applications and more computations.** So far, we have constructed the cup product, but we haven't explained why it is more useful. Now, let's try to give you some clues.

**Example 7.34.** We show that  $T^2$  and  $S^2 \vee S^1 \vee S^1$  are not homotopy equivalent. Then we know that both of them have the following cohomology groups.

$$H^0 = \mathbb{Z}, \quad H^1 = \mathbb{Z}^2, \quad H^2 = \mathbb{Z}.$$

However, we show that they are not homotopy equivalent by showing that their cup products do not coincide.

Here, it remains to see the cup product of  $S^2 \vee S^1 \vee S^1$ .

Let

$$i_{S^2} : S^2 \rightarrow X, \quad i_1 : S^1 \rightarrow X, \quad i_2 : S^1 \rightarrow X$$

be the inclusions of the three wedge summands, and let

$$p_{S^2} : X \rightarrow S^2, \quad p_1 : X \rightarrow S^1, \quad p_2 : X \rightarrow S^1$$

be the maps collapsing all other summands to the wedge point. The main idea is that all  $i^*, p^*$  are ring homomorphisms, so cup products are concentrated on each wedge summand.

Now, we explain it in detail.

Let  $u \in H^2(S^2) \cong \mathbb{Z}$  be a generator, and let  $a, b \in H^1(S^1) \cong \mathbb{Z}$  be generators for the two circle factors. Then

$$\alpha := p_1^*(a), \quad \beta := p_2^*(b), \quad \gamma := p_{S^2}^*(u)$$

generate

$$H^1(X) \cong \mathbb{Z}\alpha \oplus \mathbb{Z}\beta, \quad H^2(X) \cong \mathbb{Z}\gamma.$$

Now we claim that every cup product of two positive-degree classes in  $H^*(X)$  is zero.

First, since  $H^2(S^1) = 0$ , we have

$$\alpha \smile \alpha = p_1^*(a) \smile p_1^*(a) = p_1^*(a \smile a) = 0, \quad \beta \smile \beta = p_2^*(b) \smile p_2^*(b) = p_2^*(b \smile b) = 0.$$

Next, consider  $\alpha \smile \beta \in H^2(X)$ . Since  $H^2(X)$  is generated by  $\gamma$ , it is enough to check its restriction to each wedge summand. On the  $S^2$ -summand, both  $\alpha$  and  $\beta$  restrict to zero, because they come from the two  $S^1$ -summands. Hence

$$i_{S^2}^*(\alpha \smile \beta) = i_{S^2}^*(\alpha) \smile i_{S^2}^*(\beta) = 0.$$

On the first  $S^1$ -summand, we have  $i_1^*(\beta) = i_1^*p_2^*(b) = (p_2i)^*(b) = 0$  since  $p_2i : S^1 \rightarrow S^1$  is a constant map, so

$$i_1^*(\alpha \smile \beta) = i_1^*(\alpha) \smile i_1^*(\beta) = 0.$$

Similarly,

$$i_2^*(\alpha \smile \beta) = 0.$$

Thus  $\alpha \smile \beta = 0$ . By graded commutativity,

$$\beta \smile \alpha = -\alpha \smile \beta = 0.$$

Also, any product involving  $\gamma$  is automatically zero for degree reasons:

$$\alpha \smile \gamma = \beta \smile \gamma = \gamma \smile \alpha = \gamma \smile \beta = \gamma \smile \gamma = 0,$$

since  $H^k(X) = 0$  for  $k \geq 3$  and  $H^4(X) = 0$ .

Therefore, for  $X = S^2 \vee S^1 \vee S^1$ , all cup products of positive-degree classes vanish. In other words, the ring structure is trivial in positive degrees.

We can write directly that for  $|\alpha| = 1, |\beta| = 1, |\gamma| = 2$

$$H^*(X) \cong \mathbb{Z}[\alpha, \beta, \gamma]/(\alpha^2, \beta^2, \gamma^2, \alpha\beta, \beta\gamma, \alpha\gamma).$$

**Exercise 7.35.** In fact, the trick we use here gives the following statement: Let  $X_k$  be finitely many path-connected spaces, and  $X = \vee_k X_k$ . We set  $i_k$  as wedge component inclusions, and  $p_k$  as wedge component projections. Let  $H^+ = \bigoplus_{q>0} H^q$  as the (non-unital) subring of  $H^*$  (it has enough information since  $H^0$  record units), then show that  $i_k^*, p_k^*$  induces a ring isomorphism  $H^+(X) \cong \prod_k H^+(X_k)$ .

*Supplement material 7.36.* Here, we see that cohomology ring still cannot determine a space. However, we have the following corollary of the (highly non-trivial) theorem of Mandell [Man06]: Let  $X$  and  $Y$  be two finite CW complexes without 1-cells, suppose  $S^*(X)$  and  $S^*(Y)$  are isomorphic as  $\mathbb{E}_\infty$ -algebras, then  $X \simeq Y$ .

It is out of surprise that under some not very ridiculous conditions, the  $\mathbb{E}_\infty$ -structure of  $S^*(X)$  determines the homotopy type of  $X$ .

**Example 7.37.** Here, we explain how to compute the cup product on higher genus curves.

Let  $\Sigma$  is of genus  $g$ . Then  $H^0 \cong \mathbb{Z}, H^1 \cong \mathbb{Z}^{2g}, H^2 \cong \mathbb{Z}$ . We claim that there exists generators

$$\alpha_1, \beta_1, \dots, \alpha_g, \beta_g \in H^1(\Sigma)$$

and a generator  $\omega \in H^2(\Sigma)$ , such that

$$\alpha_i \smile \beta_j = \delta_{ij}\omega, \quad \beta_i \smile \alpha_j = -\delta_{ij}\omega,$$

for all  $i = 1, \dots, g$ , and for all  $i, j = 1, \dots, g$ ,

$$\alpha_i \smile \alpha_j = \beta_i \smile \beta_j = 0.$$

Let

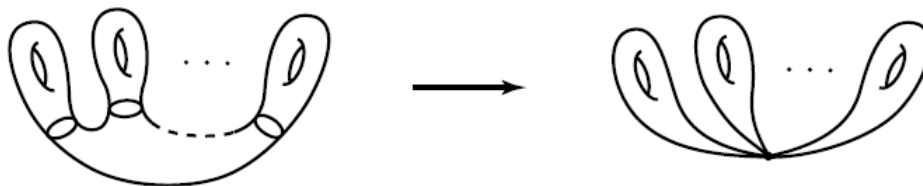
$$X = T_1 \vee \dots \vee T_g$$

be a wedge sum of  $g$  copies of the torus  $T_i \cong S^1 \times S^1$ . By collapsing the complement of the  $g$  handles of  $\Sigma$  to the wedge point, we obtain a quotient map

$$q : \Sigma \rightarrow X, \quad p_i : X \rightarrow T_i.$$

For each  $i$ , let  $H^1(T_i) = \mathbb{Z}x_i \oplus \mathbb{Z}y_i$  be the generators, and let  $u_i \in H^2(T_i)$  be the generator such that

$$x_i \smile y_i = u_i, \quad y_i \smile x_i = -u_i, \quad x_i \smile x_i = y_i \smile y_i = 0.$$



Genus  $g$  surface collapsing into wedge of torus. Picture from Hatcher.

We also regard these classes as classes on  $X$  via the inclusions

$$p_i^* : H^*(T_i) \hookrightarrow H^*(X).$$

Since  $X$  is a wedge sum, all cup products between positive-degree classes coming from different wedge summands vanish. Hence for  $i \neq j$  we have

$$x_i \smile x_j = 0, \quad y_i \smile y_j = 0, \quad x_i \smile y_j = 0.$$

Now define

$$\alpha_i := q^*(x_i), \quad \beta_i := q^*(y_i) \in H^1(\Sigma).$$

Since  $q$  identifies the 1-skeleton of  $\Sigma$  with the wedge of the  $2g$  circles underlying the  $g$  tori, the classes  $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$  form a basis of  $H^1(\Sigma)$ .

Next, let  $[\Sigma] \in H_2(\Sigma)$  be a generator. Under the map  $q_*$ , we have

$$q_*[\Sigma] = \pm u_1 + \dots + \pm u_g \in H_2(X) \cong \bigoplus H_2(T_i).$$

By rearrange the sign of  $x_i, y_i, u_i$ , we may assume that for every  $i$ ,

$$\langle q^* u_i, [\Sigma] \rangle = \langle u_i, q_*[\Sigma] \rangle = 1,$$

i.e.  $q_*[\Sigma] = u_1 + \dots + u_g$ . Since  $H^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z}$ , it follows that all  $q^* u_i$  are equal to the same generator  $\omega \in H^2(\Sigma; \mathbb{Z})$ .

Hence

$$\alpha_i \smile \beta_i = q^*(x_i \smile y_i) = q^*(u_i) = \omega, \quad \beta_i \smile \alpha_i = q^*(y_i \smile x_i) = -q^*(u_i) = -\omega,$$

and for  $i \neq j$ ,  $\alpha_i \smile \beta_j = 0$ .

Similarly,

$$\alpha_i \smile \alpha_j = q^*(x_i \smile x_j) = 0, \quad \alpha_i \smile \alpha_i = \beta_i \smile \beta_j = \beta_i \smile \beta_i = 0.$$

The following discussion will assume you know differential forms and de Rham theorem; feel free to skip it.

**Example 7.38.** Here, in terms of de Rham theorem (though we didn't completely prove it), we compute  $H^*(\mathbb{C}P^n; \mathbb{R}) \simeq H_{dR}^*(\mathbb{C}P^n)$ . We refer to [Bre93, Section 3.9] for more details.

There exists a differential 2-form called the Fubini-Study form  $\omega$ , which is closed and non-exact since it is a Volume form on the compact manifold  $\mathbb{C}P^1$  (you can do this by direct computation). In general,  $\omega^k$  are also closed and non-exact since they are a volume form on  $\mathbb{C}P^k$  (in that case, you can do it by Calibration theory). Then all of  $[\omega]^k$  generates  $H_{dR}^{2k} \mathbb{C}P^n$  for  $k = 0, 1, \dots, n$ . Moreover, by degree reason, we have  $[\omega]^{n+1} = 0$ .

Therefore, we have  $H^*(\mathbb{C}P^n; \mathbb{R}) \cong H_{dR}^*(\mathbb{C}P^n) \cong \mathbb{R}[x]/(x^{n+1})$  for  $x = [\omega]$ .

*Remark 7.39.* We also have  $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$ . The main non-trivial ingredient here is to show that  $x^k$  are generators for the generator  $x$  of  $H^2(\mathbb{C}P^n)$  (where we use some non-trivial integral computation in the de Rham theory). We will give a proof using the Poincaré duality Example 8.38.

**Exercise 7.40.** \* Show that if  $M$  is a compact symplectic manifold, then  $H^{2k}(M)$  are non-trivial for  $k = 0, 1, \dots, \dim M$ .

## 8. MANIFOLD AND POINCARÉ DUALITY

In this section, we study topological manifolds. Importantly, we explain the Poincaré duality.

## 8.1. Poincaré duality.

**Definition 8.1.** A Hausdorff and second countable topological space  $M$  is called a topological  $n$ -manifold if for every  $x \in M$ , there exists an open neighborhood  $U \subset M$  of  $x$  and a homeomorphism

$$\varphi : U \xrightarrow{\sim} V$$

onto an open subset  $V \subset \mathbb{R}^n$ .

Such a pair  $(U, \varphi)$  is called a chart around  $x$ .

*Remark 8.2.* We only consider manifolds without boundary here. You may consult for references for the definition of manifolds with boundary (or with corner in general).

**Example 8.3.** The spaces  $\mathbb{R}^n$ ,  $S^n$ , and every open subset of  $\mathbb{R}^n$  are topological  $n$ -manifolds.

Closed surface is a topological 2-manifold (in fact, this is the actual definition for surfaces).

Recall that in Remark 4.39, we already saw that local homology should be useful for manifolds. Now we prove it.

**Proposition 8.4.** *Let  $M$  be a topological  $n$ -manifold and  $x \in M$ , and a commutative ring  $R$ . Let  $j_x : M \setminus \{x\} \hookrightarrow M$  be the open inclusion, then*

$$H_n(C(j_x); R) \cong R, \quad H_q(C(j_x); R) \cong 0, \quad \forall q \neq n.$$

*Proof.* After picking a chart  $\varphi : U \xrightarrow{\sim} V \subset \mathbb{R}^n$  with  $x \in U$ . By shrinking  $U$  if necessary, we may assume that there exists an open ball  $B \subset V$  centered at  $\varphi(x)$  such that  $\overline{B} \subset V$ . Set

$$D := \varphi^{-1}(\overline{B}) \subset U.$$

Then  $D \cong D^n$  and  $x \in \text{Int}(D)$ .

Then all the rest follows from the same excision argument as Proposition 4.38.  $\square$

*Supplement material 8.5.* Topological spaces that satisfy the conclusion of Proposition 8.4 are called homology manifolds, and then the proposition can be interpreted by: Topological manifolds are homology manifolds.

However, there exist homology manifolds that are not topological manifolds: Take  $P$  to be the homology  $n$ -sphere (it is misleading that a homology  $n$ -sphere is a topological manifold), which is a topological manifold that satisfies  $H_q(P) \cong H_q(S^n)$ . Then its suspension  $\Sigma P$  is a homology manifold. But,  $\Sigma P$  is not a topological manifold except  $P = S^n$ .

There indeed exists a homology  $n$ -sphere  $P$  that is not  $S^n$ , the most famous examples come from Poincaré. The Poincaré duality exactly comes from his study of those homology spheres. The study eventually leads to the Poincaré conjecture, which was solved by Perelman, which claims that a homology 3-sphere  $P$  is literally a sphere when  $\pi_1(P)$  is trivial.

However, another interesting side of the story is that the double suspension theorem of Edwards–Cannon tells  $\Sigma^2 P \cong S^{n+2}$ , which is a topological manifold.

**Exercise 8.6.** \* Show that if  $P$  is a homology  $n$ -sphere, i.e. is a topological manifold that satisfies  $H_q(P) \cong H_q(S^n)$ , then for  $M = \Sigma P$ , the conclusion of Proposition 8.4 is true. I.e.  $\Sigma P$  is a homology manifold.

On the other hand, show that  $M = \Sigma P$  is a topological manifold if  $P \cong S^n$ , and if  $M = \Sigma P$  is a topological manifold, then  $P \simeq S^n$  (here, you may want to conclude  $P \cong S^n$ , this is the content of the generalized Poincaré conjecture, which is true, but more difficult to prove).

**Definition 8.7.**  $M$  be a topological  $n$ -manifold and  $x \in M$ . A generator

$$\mu_x \in H_n(C(j_x); R) \cong R$$

is called a local  $R$ -orientation of  $M$  at  $x$ .

*Remark 8.8.* By Proposition 8.4, at each point there are exactly two choices of local  $\mathbb{Z}$ -orientations, namely  $\mu_x$  and  $-\mu_x$ . If  $R$  is of characteristic 2, then only one choice of local  $R$ -orientation.

Now, we ask whether one can choose local orientations coherently.

**Definition 8.9.** Let  $M$  be a topological  $n$ -manifold and  $R$  be a unital commutative ring. An  $R$ -orientation of  $M$  is a map

$$M \rightarrow \bigsqcup_{x \in M} H_n(C(j_x); R), \quad x \mapsto \mu_x,$$

such that it is coherent in the sense: for every  $x \in M$ , there exists an open neighborhood  $j_U : U \rightarrow M$  of  $x$  and a class

$$\mu_U \in H_n(C(j_U); R)$$

whose image under

$$H_n(C(j_U); R) \rightarrow H_n(C(j_y); R)$$

is  $\mu_y$  for every  $y \in U$ .

If  $M$  admits an  $R$ -orientation, then we call  $M$   $R$ -orientable. We simply say orientable/orientation if  $R = \mathbb{Z}$ .

*Remark 8.10.* The notion is compatible with the notion of orientation for smooth manifold. However, we need more effort to prove the coincidence. Let's omit it here.

**Example 8.11.** The space  $\mathbb{R}^n$  is orientable. For every  $x$ , we define the translation map  $T_x(v) = v + x$ .

We pick  $\mu \in H_n(C(j_0)) \cong \mathbb{Z}$ , and then define the

$$\mu_x := (T_x)_n(\mu) \in H_n(C(j_x)),$$

which is a generator since  $T_x$  are homeomorphisms for all  $x$ .

To show the compatibility, at  $0 \in \mathbb{R}^n$ , we pick  $E = E^n$  be an open ball of radius 1 centered at 0, then  $H_n(C(j_E)) \cong \mathbb{Z}$  and by naturality of excision we can pick a generator  $\mu_E$  such that the image of  $\mu_E$  under

$$H_n(C(j_E)) \rightarrow H_n(C(j_0))$$

is  $\mu_0$ .

We set  $U_0 = E$ ,  $U_x = T_x(U_0)$ , and

$$\mu_{U_x} = (T_x)_n(\mu_E) : H_n(C(j_{U_0})) \xrightarrow{\cong} H_n(C(j_{U_x})).$$

**Exercise 8.12.** Check the every  $y \in U_x$ , the image of  $\mu_{U_x}$  under

$$H_n(C(j_{U_x})) \rightarrow H_n(C(j_y))$$

is exactly  $\mu_y$ . And then conclude that  $\mathbb{R}^n$  is  $\mathbb{Z}$ -orientable.

**Exercise 8.13.** Show that  $S^n$  is  $\mathbb{Z}$ -orientable.

*Remark 8.14.* If  $M$  is connected and orientable, then  $M$  has exactly two  $\mathbb{Z}$ -orientations.

**Exercise 8.15.** Using the universal coefficient theorem Theorem 7.12 (in fact, the pair version, which is still true following the original argument) to show that if  $M$  is  $\mathbb{Z}$ -orientable, then  $M$  is  $R$ -orientable for every commutative ring  $R$ .

**Exercise 8.16.** For commutative ring  $R$  is characteristic 2, for example  $\mathbb{F}_2$ , show that any manifolds are  $R$ -orientable. Hint: point here is  $\mu_x = -\mu_x$ , you have no choice for local-orientation.

The following theorem is pretty fundamental:

**Theorem 8.17.** Let  $M$  be a connected compact topological  $n$ -manifold.

(1) If  $M$  is  $R$ -orientable, then  $H_n(M; R) \rightarrow H_n(C(j_x); R) \cong R$  is an isomorphism for every  $x \in M$ .

(2) If  $M$  is not  $R$ -orientable, then  $H_n(M; R) \rightarrow H_n(C(j_x); R) \cong R$  is injective with image  $\{r \mid 2r = 0\}$  for every  $x \in M$ .

(3)  $H_q(M; R) = 0$  for  $q > n$ .

In the other word, the condition  $H_n(M; R) \cong R$  is equivalent to  $M$  is  $R$ -orientable, and a choice of generator for  $H_n(M; R) \cong R$  is equivalent to give an  $R$ -orientation of  $M$ .

*Remark 8.18.* Its proof is long enough. As we will omit many proofs in this section, it is fine to start from this one. You may consult [Hat02, Theorem 3.26]. Notice that in Hatcher, he actually states and proves the result for  $H_n(M, M \setminus \{x\}; R)$  in the place of  $H_n(C(j_x))$ . If you know what later is, you shall know that they are naturally isomorphic.

**Example 8.19.** By this theorem, we know that the genus  $g$  surface  $\Sigma$ , the complex projective space  $\mathbb{C}P^n$  are orientable. And the real projective space  $\mathbb{R}P^n$  is orientable if and only if  $n$  is odd.

**Definition 8.20.** Let  $M$  be a connected  $R$ -oriented (here, it means that we fix one orientation already) compact  $n$ -manifold. Then the class

$$[M] \in H_n(M; R),$$

whose image in  $H_n(C(j_{U_x}); R)$  is  $\mu_x$ , is called the fundamental class of  $M$  (of the given orientation.)

**Exercise 8.21.** Let  $M$  be a connected compact  $n$ -manifold that is  $R$ -orientable. Show that if  $[M]$  is a fundamental class of one orientation, then  $-[M]$  is the fundamental class of another orientation (call the opposite orientation).

To compact manifolds, we have the following technical result that is very useful.

**Theorem 8.22.** A compact topological manifold  $X$  of dimensional not equals to 4 is homotopy equivalent to a finite CW complex. Consequently,  $H_q(X)$  and  $H^q(X)$  are finitely generated abelian groups for all  $q$ . So  $b_q(X)$  are well-defined for compact manifolds.

**Exercise 8.23.** Show that if  $\dim X = 4$ , then  $H_q(X)$  and  $H^q(X)$  are finitely generated abelian groups for all  $q$ . Hint: Consider  $X \times S^1$ , and show it is a compact topological manifold of dimension 5. Then you can use the Künneth formula.

*Supplement material 8.24.* There are some subtly here. When  $X$  is a non-compact manifold, then  $X$  is homotopy equivalent to a CW complex that has at most countable  $q$ -cells for every  $q \leq \dim X$ . So, the only unknown case will be compact topological manifold of  $\dim = 4$  (anyway, we know finiteness for (co)homology by the exercise). In this case, the good story is that if  $X$  is compact smooth manifold, then we know  $X$  is homotopy equivalent to a finite CW complex. It is more subtle to ask  $\Delta$ -complex structure (triangulation), it is proven by Manolescu using Seiberg-Witten theory that there exists for every  $n \geq 5$  an topological  $n$ -manifold that do not have a  $\Delta$ -complex structure. Meanwhile, all smooth manifolds admit a  $\Delta$ -complex structure.

**Exercise 8.25.** If  $M$  is a connected compact topological  $n$ -manifold, then if  $M$  is orientable, then  $H_{n-1}(M)$  (as a finitely generated abelian group) is free, and if  $M$  is not orientable, then  $H_{n-1}(M) \cong \mathbb{Z}' \oplus \mathbb{Z}/2$ . Hint: use the universal coefficient theorem Theorem 2.28.

Consequently, by the universal coefficient theorem Theorem 7.12, we have  $H^n(M) \cong \mathbb{Z}$  when  $M$  is orientable and  $H^n(M) \cong \mathbb{Z}/2$  when  $M$  is non-orientable. Then  $M$  is orientable if and only if  $H^n(M) \cong \mathbb{Z}$ , and when  $M$  is orientable, there exists a unique element in  $H^n(M)$  corresponding  $[M] \in H_n(M)$ , which is still denoted by  $[M] \in H^n(M)$  sometime, and we call it (cohomological) fundamental class.

**Exercise 8.26.** Previously, we construct an explicit generator for  $H_n(S^n)$ , say  $[\sigma_{S^n}]$ , see Exercise 4.18. Show that it is the fundamental class for an orientation of  $S^n$ .

*Supplement material 8.27.* In the example for sphere, where we put a  $\Delta$ -complex structure, we see its fundamental class can be computed by a suitable sum of all top dimension singular simplexes of the  $\Delta$ -complex structure.

The construction works in general: Suppose a compact oriented topological  $n$ -manifold  $M$  has a  $\Delta$ -complex structure (then it must be finite), we can construct the following  $n$ -chain

$$\sigma_M = \sum_{\alpha} \varepsilon_{\alpha} \varphi_{\alpha}^n,$$

where  $\varepsilon_{\alpha} = \pm 1$  is determined in the following way: we take  $\varepsilon_{\alpha} = 1$  (resp.  $-1$ ) if for some  $x \in \varphi_{\alpha}^n(\text{Int}(\Delta^n))$  we have  $[\varphi_{\alpha}^n] \in H_n(C(j_x))$  coincides (resp. opposite) with the local orientation  $\mu_x$  determined by the orientation of  $M$ .

Then one can check that  $\partial\sigma_M = 0$  and  $[\sigma_M] = [M]$  for the given orientation.

It explains the strange sign in Exercise 2.18, and you may also see from this construction why any manifold that equip with a  $\Delta$ -complex structure is  $\mathbb{F}_2$ -oriented. Another such an example can be found in Example 8.34.

This combinatorial approach is closer to the original approach of Poincaré on his duality theorem.

*Supplement material 8.28.* Fundamental class also provides fruitful examples of homology class in manifolds: If  $i : Y \rightarrow M$  is a submanifold of a manifold  $M$ , and  $Y$  is compact oriented,  $i_*[Y] \in H_{\dim Y}(M)$  is a homology class in  $M$ , and very often to be non-trivial. It is interesting to know if every homology class in  $H_q(M)$  can be realized by a submanifold class in this way, which is known to be false in general (some examples are given by Thom using the Steenrod square [Tho54]).

There are many further discussions here: 1) It is proven by Thom [Tho54] that for all homology classes  $z$ , there exists  $l \in \mathbb{N}$  such that  $lz$  is realized by a submanifold  $i_*[Y] = lz$ . 2) For every homology class, Zinger [Zin08] proves that one can realize it by a pseudo-cycle, which has application in differential geometry definition for the virtual fundamental class of Gromov-Witten invariant. 3) In the case of projective varieties, a similar question about algebraic cycles is known as the Hodge conjecture, which is still widely open.

Recall the results in Exercise 7.26, which we combine with Definition 8.20 to formula the following theorem

**Theorem 8.29** (Poincaré duality). *Let  $M$  be a connected closed  $R$ -oriented topological  $n$ -manifold. Then cap product with the fundamental class induces an isomorphism*

$$PD : H^q(M; R) \xrightarrow{\sim} H_{n-q}(M; R), \quad PD(c) := c \frown [M]$$

for every  $q$ .

*Idea of the proof.* Again, we will not prove it. See [Hat02, Section 3.3].

The idea for the proof is that both sides of the equivalence form cosheaves on  $M$ ; and to check they are isomorphic cosheaves, you only need to compare their costalks, which can be easily seen from homology/cohomology of small balls.

Also notice that in Hatcher, he proves the result using  $H_n(M, M \setminus K; R)$  where  $K \subset M$  is compact, you can safely replace run his argument for  $H_n(C(M \setminus K \hookrightarrow M))$ .  $\square$

*Supplement material 8.30.* The modern form of Poincaré duality is much much far from the result. One version is the Verdier duality, which you might learn in the sheaf theory. Another one is Lurie's covariant Verdier duality, which states an equivalence between the sheaf category and the cosheaf category under mild conditions.

**Exercise 8.31.** \* If  $M$  is a compact oriented manifold with a  $\Delta$ -complex structure, and we take the fundamental cycle  $\sigma_M$  constructed in Supplement material 8.27. We set  $[f_{i_0}, \dots, f_{i_k}]_{\alpha} : \Delta^k \xrightarrow{[v_{i_0}, \dots, v_{i_k}]} \Delta^n \xrightarrow{\varphi_{\alpha}^n} X$ , then show that

$$PD([c]) = [c] \cap [\sigma_M] = \sum_{\alpha} c([f_0, \dots, f_q]_{\alpha}) [f_q, \dots, f_n]_{\alpha}.$$

We will see a precise example for the computation in Example 8.34.

You may draw some picture to see the relation of the formula and dual graph (in the graph theory sense).

8.2. Examples and applications.

**Exercise 8.32.** Let  $M$  be a connected closed oriented  $n$ -manifold. Recall that we can define the Betti number  $b_q(X) = b^q(X)$  as the rank of the corresponding homology or cohomology (See Theorem 8.22, Exercise 7.14).

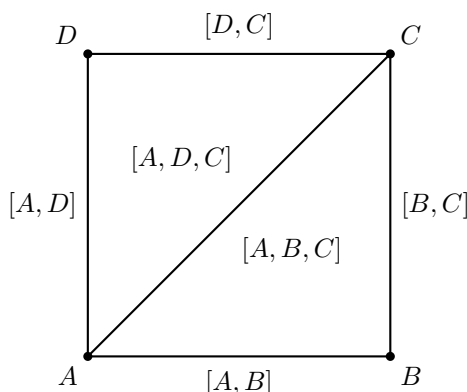
Show that 1)  $b_q(M) = b_{n-q}(M)$  for all  $q$ ; 2) If  $M$  is odd dimension, then  $\chi(M) = 0$ .

One interesting application is that you can use Poincaré duality to distinguish manifolds.

**Exercise 8.33.** Show that  $S^2 \vee S^2$  is not a manifold. Hint: Use  $\mathbb{F}_2$ -coefficient and apply the Poincaré duality.

**Example 8.34.** Here, let us compute the  $PD : H^1(T^2) \rightarrow H_1(T^2)$ .

As in Example 4.32, we think  $T^2$  as the quotient of  $[0, 1]^2$  by identify certain edges, we denote  $q : [0, 1]^2 \rightarrow T^2$ :



We set  $p_1, p_2 : T^2 = S^1 \times S^1 \rightarrow S^1$  projections. Let  $u \in H^1(S^1)$  be a generator, and set

$$\alpha := p_1^*(u), \quad \beta := p_2^*(u) \in H^1(T^2).$$

We know they are generators of  $H^1(T^2)$ .

Let  $i_1 : S^1 \rightarrow T^2, t \mapsto (t, 0)$ ,  $i_2 : S^1 \rightarrow T^2, t \mapsto (0, t)$ , and a  $[S^1]$  be the fundamental class of  $S^1$ , then

$$a = (i_1)_*([S^1]) = [S^1 \times \{*\}], \quad b = (i_2)_*([S^1]) = [\{*\} \times S^1] \in H_1(T^2),$$

then, after choosing the fundamental class  $[T^2] \in H_2(T^2)$  as below, we will show that

$$\alpha \frown [T^2] = b, \quad \beta \frown [T^2] = -a.$$

We compute directly from the chain-level definition of cap product given in Exercise 7.26, especially, the compatibility

$$\langle c \smile d, \sigma \rangle = \langle c, d \frown \sigma \rangle.$$

Write

$$A = (0, 0), \quad B = (1, 0), \quad C = (1, 1), \quad D = (0, 1).$$

Consider the two singular 2-simplices

$$\sigma_U := q \circ [A, B, C], \quad \sigma_L := q \circ [A, D, C],$$

and define

$$z := \sigma_U - \sigma_L \in S_2(T^2).$$

We claim that  $z$  is a 2-cycle by direct computation.

Indeed,

$$\partial \sigma_U = q_{\#}([B, C] - [A, C] + [A, B]), \quad \partial \sigma_L = q_{\#}([D, C] - [A, C] + [A, D])$$

Hence

$$\partial z = q_{\#}([B, C] - [D, C] + [A, B] - [A, D]).$$

But in the quotient  $T^2$  we have

$$q_{\#}[B, C] = q_{\#}[A, D], \quad q_{\#}[D, C] = q_{\#}[A, B],$$

so  $\partial z = 0$ . Moreover, the cycle represent the 2-cell  $I^2$ , then we know  $[z] \in H_2(T^2)$  is a generator, and we can find an orientation to make it a fundamental class  $[z] = [T^2]$ .

Now define 1-cycles

$$a := q_{\#}[A, B], \quad b := q_{\#}[A, D].$$

These represent the two generators of  $H_1(T^2)$  coming from the two  $S^1$ -factors.

Next we evaluate  $\alpha$  and  $\beta$  on the relevant edges. Since

$$\alpha = p_1^*(u), \quad \beta = p_2^*(u),$$

we have

$$\alpha([A, B]) = 1, \quad \alpha([A, D]) = 0, \quad \alpha([A, C]) = 1,$$

and

$$\beta([A, B]) = 0, \quad \beta([A, D]) = 1, \quad \beta([A, C]) = 1.$$

Geometrically,  $\alpha$  measures winding in the first factor, while  $\beta$  measures winding in the second factor.

We now compute the cap products. Since  $\alpha, \beta \in S^1(T^2)$  and  $z \in S_2(T^2)$ , for a singular 2-simplex  $\sigma$  we have

$$c \frown \sigma = c(1\sigma) \sigma_1.$$

First, for  $\alpha$ :

$$1\sigma_U = [A, B], \quad (\sigma_U)_1 = [B, C],$$

hence

$$\alpha \frown \sigma_U = \alpha([A, B])[B, C] = [B, C].$$

Also,

$$1\sigma_L = [A, D], \quad (\sigma_L)_1 = [D, C],$$

so

$$\alpha \frown \sigma_L = \alpha([A, D])[D, C] = 0.$$

Therefore

$$\alpha \frown z = [B, C].$$

In  $T^2$ , the right vertical edge  $[B, C]$  is identified with the left vertical edge  $[A, D]$ , so

$$[\alpha \frown z] = b.$$

Thus

$$\alpha \frown [z] = b.$$

Next, for  $\beta$ : similarly, we have

$$\beta \frown z = 0 - [D, C].$$

Since the top horizontal edge  $[D, C]$  is identified with the bottom horizontal edge  $[A, B]$ , we get

$$[\beta \frown z] = -a.$$

Thus

$$\beta \frown [z] = -a.$$

So, with the orientation represented by the cycle  $z = \sigma_U - \sigma_L$ , we obtain

$$\alpha \frown [T^2] = b, \quad \beta \frown [T^2] = -a.$$

**Exercise 8.35.** Compare to Example 7.37, compute  $PD : H^1(\Sigma) \rightarrow H_1(\Sigma)$  for genus  $g$  surface using above computation Example 8.34: For the generator  $\alpha_i, \beta_i$  of  $H^1(\Sigma)$ , we have  $PD(\alpha_i) = b_i$  and  $PD(\beta_i) = -a_i$ , where  $a_i, b_i$  represent corresponding cycles in the gluing diagram in Exercise 4.34. You may also need Exercise 7.26-(4).

Our computation motivates the following definition.

**Exercise 8.36.** Let  $M$  be an compact oriented topological  $n$ -manifold. We define the following pairing, which is still denoted by  $\smile$ ,

$$\smile: H^{n-p}(M) \times H^p(M) \rightarrow \mathbb{Z}, \quad (a, b) \mapsto \langle a \smile b, [M] \rangle.$$

On the other hand, in Exercise 7.11, we have that the dual pairing

$$\kappa: H^p(X) \otimes H_p(X) \rightarrow \mathbb{Z},$$

which is generally not perfect due to the universal coefficient theorem.

(1) Show that we have the following commutative diagram

$$\begin{array}{ccc} H^p(M) \times H^{n-p}(M) & \xrightarrow{\smile} & \mathbb{Z} \\ \downarrow \text{id} \otimes PD & & \downarrow \\ H^p(M) \times H_p(M) & \xrightarrow{\kappa} & \mathbb{Z} \end{array}$$

(2) Show the pairing  $\smile$  induces a perfect pairing

$$H^p(M)_{free} \times H^{n-p}(M)_{free} \rightarrow \mathbb{Z}, \quad (a, b) \mapsto \langle a \smile b, [M] \rangle,$$

and then where  $A_{free}$  means the free part of the finitely generated abelian group  $A$ .

Similarly, we have for a field  $\mathbb{F}$

$$H^p(M; \mathbb{F}) \times H^{n-p}(M; \mathbb{F}) \rightarrow \mathbb{F}, \quad (a, b) \mapsto \langle a \smile b, [M] \rangle$$

is perfect.

Hint: Use the universal coefficient theorem Theorem 7.12 and Exercise 7.26-(3).

(3) Show that for a  $2n$ -dimensional manifold  $M$ , we have  $\smile: H^n(M) \times H^n(M) \rightarrow \mathbb{Z}$  is symmetric if  $2n = 4k$  and anti-symmetric if  $2n = 4k + 2$ .

For example, for the genus  $g$  curve, the matrix of the pairing under the bases we constructed in Example 7.37 is  $\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ .

(4) \* For a  $2n = 4k + 2$  dimensional compact oriented manifold  $M$ , we have  $\chi(M)$  is even. Notice that this is NOT true for non-oriented manifolds, for example  $\chi(\mathbb{R}P^2) = 1$ .

*Supplement material 8.37.* To certain manifolds, the pairing  $\smile$  is crucial. For example, Freedman show that for simply connected 4-dimensional topological manifolds (simply connected implies orientability), their homeomorphism class is determined by the pairing  $\smile$ .

**Example 8.38.** We already know that  $\mathbb{C}P^n$  is a  $2n$ -dimension manifold, and  $H_{2n}(\mathbb{C}P^n) \cong \mathbb{Z}$  shows that  $\mathbb{C}P^n$  is orientable, so we can use Poincaré duality for it.

Here, we show the cohomology ring of  $H^*(\mathbb{C}P^n)$  is isomorphic to  $\mathbb{Z}[x]/(x^{n+1})$  using the Poincaré duality, and moreover we can pick  $x$  such that have  $PD(x^n) = [\text{pt}]$ . We will actually compute the Poincaré duality  $PD(x^k)$  for  $k = 0, \dots, n$ .

We prove it by induction. When  $n = 1$ , we know  $\mathbb{C}P^1 \cong S^2$ , and then  $H^*(\mathbb{C}P^1) = H^*(S^2) \cong \mathbb{Z}[x]/(x^2)$  for  $|x| = 2$  by Example 7.22 and we pick orientation of  $S^2$  such that  $PD(x) = [p]$ .

Now, consider the higher dimension.

As the induction hypothesis, we assume  $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$  with  $PD(x^n) = x^n \smile [\mathbb{C}P^n] = [\text{pt}]$ .

We consider the (hyperplane) embedding

$$i: \mathbb{C}P^n \rightarrow \mathbb{C}P^{n+1}, \quad [z_0, \dots, z_n] \rightarrow [z_0, \dots, z_n, 0].$$

Then we have a ring homomorphism

$$i^*: H^*(\mathbb{C}P^{n+1}) \rightarrow H^*(\mathbb{C}P^n),$$

and by the cellular homology computation, or the dual version of mapping cone sequence computation, we know that  $i^q$  is a group isomorphism for  $0 \leq q \leq 2n$ . In particular, we may write  $H^{2k}(\mathbb{C}P^{n+1}) = \mathbb{Z}[y^k]$  for  $0 \leq k \leq n$  for  $x^k = i^*(y^k)$ , since  $x^k$  is a generator of  $H^{2k}(\mathbb{C}P^n)$  by the induction hypothesis.

Now, by Exercise 7.26-(2), we have

$$\langle y^{n+1}, [\mathbb{C}P^{n+1}] \rangle = \langle y^n, y \frown [\mathbb{C}P^{n+1}] \rangle = \langle y^n, PD(y) \rangle.$$

(The non trivial place:) In this formula, we have that  $PD(y) \in H_{2n}(\mathbb{C}P^{n+1})$  is a generator since  $PD : H^2 \rightarrow H_{2n}$  is an isomorphism between abelian groups.

On the other hand, we know  $y^n$  is a generator of  $H^{2n}(\mathbb{C}P^{n+1})$ . Therefore, by Exercise 8.36-(2), the pairing

$$\langle y^{n+1}, [\mathbb{C}P^{n+1}] \rangle = \langle y^n, PD(y) \rangle = \pm 1,$$

and then we have  $y^{n+1}$  is a generator of  $H^{2n+2}(\mathbb{C}P^{n+1})$ . Moreover, we can always assume it equals 1 by taking a suitable orientation  $[\mathbb{C}P^{n+1}]$ .

Consequently, we have

$$H^*(\mathbb{C}P^{n+1}) \cong \mathbb{Z}[y]/(y^{n+2}), \quad \langle 1, y^{n+1} \frown [\mathbb{C}P^{n+1}] \rangle = \langle y^{n+1}, [\mathbb{C}P^{n+1}] \rangle = 1$$

which finish the induction.

Now, we compute the Poincaré duality (go back to  $\mathbb{C}P^n$  and  $\mathbb{Z}[x]/(x^{n+1})$ ).

Similar to the previous  $i$ , we set the subspace embedding to be  $i_k : \mathbb{C}P^{n-k} \rightarrow \mathbb{C}P^n$ ,  $[z_0, \dots, z_{n-k}] \rightarrow [z_0, \dots, z_{n-k}, \dots, 0]$ . We claim that  $PD(x^k) = (i_k)_*[\mathbb{C}P^{n-k}]$ .

Indeed, both classes lie in  $H_{2n-2k}(\mathbb{C}P^n) \cong \mathbb{Z}$ , so it is enough show their pairing with the generator  $x^{n-k} \in H^{2n-2k}(\mathbb{C}P^n)$  are both 1: On one hand, we have

$$\langle x^{n-k}, PD(x^k) \rangle = \langle x^{n-k}, x^k \frown [\mathbb{C}P^n] \rangle = \langle x^n, [\mathbb{C}P^n] \rangle = 1,$$

on the other hand, we have

$$\langle x^{n-k}, (i_k)_*[\mathbb{C}P^{n-k}] \rangle = \langle i_k^*(x^{n-k}), [\mathbb{C}P^{n-k}] \rangle = \langle x^{n-k}, [\mathbb{C}P^{n-k}] \rangle = 1.$$

Therefore, we have

$$PD(x^k) = (i_k)_*[\mathbb{C}P^{n-k}],$$

as claimed.

*Remark 8.39.* Here, try to compare Example 7.38 as explained in Remark 7.39.

Notice that we prove this fact in  $\mathbb{R}$ -coefficient using a certain integration trick, which indicates the deep relation between Poincaré duality and integration of differential forms. In fact, we can develop the entire de Rham version of Poincaré duality using integration, this is the topic of [BT82].

**Exercise 8.40.** \* Show that  $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x]$  with  $|x| = 2$ . The ring is useful when you study the first Chern class and the  $S^1$ -equivariant cohomology.

**Exercise 8.41.** Using the  $H^*(\mathbb{C}P^{2n}) \cong \mathbb{Z}[x]/(x^{2n+1})$  to show that there is no homotopy equivalence  $f : \mathbb{C}P^{2n} \rightarrow \mathbb{C}P^{2n}$  such that  $f^{4n}$  maps  $x^{2n}$  to  $-x^{2n}$ .

Here, we notice that  $x^{2n} = [\mathbb{C}P^{2n}]$  (the RHS means the cohomological fundamental class here) for an orientation. Then the exercise tells that all homotopy equivalences of  $\mathbb{C}P^{2n}$  preserve the orientation of  $\mathbb{C}P^{2n}$ .

**Exercise 8.42.** Now, you may be strong enough to prove by yourself the ring isomorphism  $H^*(\mathbb{R}P^n, \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$  with  $|x| = 1$ .

*Supplement material 8.43.* We mention a dual point of view for the pairing (in the language of differential topology). Let  $M$  be an oriented compact  $n$ -dimensional smooth manifold. We define the following pairing, which is called the intersection product,

$$\cap : H_{n-p}(M) \otimes H_{n-q}(M) \rightarrow H_{n-p-q}(M)$$

by the following diagram

$$\begin{array}{ccc} H^p(M) \times H^q(M) & \xrightarrow{\smile} & H^{p+q}(M) \\ \downarrow PD \otimes PD & & \downarrow PD \\ H_{n-p}(M) \times H_{n-q}(M) & \xrightarrow{\cap} & H_{n-p-q}(M) \end{array}$$

The interesting thing is that for chains represented by submanifolds, we can compute the intersection product by geometric intersection (counted algebraically):

Let  $i : X \rightarrow M$ ,  $j : Y \rightarrow M$  are oriented compact smooth submanifolds. We want to compute  $i_*[X] \cap j_*[Y]$  (the intersection product, not the set theoretical intersection).

We assume that  $di_p \oplus dj_p : T_pX \oplus T_pY \rightarrow T_pM$  is surjective for all  $p \in X \cap Y$ , i.e.  $X$  and  $Y$  intersect transversely. This is an gentle assumption: By Thom's transversality theory, we can always pick a small perturbation of  $i, j$  such that the transversality condition is satisfied and the cycles  $i_*[X], j_*[Y]$  do not change.

In this case, we have that  $k : X \cap Y \rightarrow M$  is a submanifold of dimension  $n - \dim X - \dim Y$ , and there exists a unique orientation such that  $T_pM \cong T_pX \oplus T_pY / T_p(X \cap Y)$  is orientation preserving, and we take the class  $k_*[X \cap Y]$  for this orientation. We define  $i_*[X] \cap j_*[Y] := k_*[X \cap Y]$ . The main theorem here is that  $PD^{-1}(i_*[X] \cap j_*[Y]) = PD^{-1}(i_*[X]) \smile PD^{-1}(j_*[Y])$  [Bre93, Chapter IV, Theorem 11.9]. It gives a geometric interpretation of the cup product.

For example, when  $\dim X + \dim Y = n$ , transversality means that  $\forall p \in X \cap Y$  the linear map  $J_p = di_p \oplus dj_p$  is rank  $n$ , and in this situation  $X \cap Y$  is a finite set. We set  $\epsilon_p = \text{sign}(\det(J_p))$ , then we have

$$i_*[X] \cap j_*[Y] = \sum_{p \in X \cap Y} \epsilon_p [\text{pt}] \in \mathbb{Z} \cong H_0(M).$$

For example in  $T^2$  (compare to Example 8.34), you can see that  $PD(\alpha \smile \beta) = b \cap (-a) \in H_0(T^2)$ , and  $a, b$  are represented by some  $S^1$ . Those circles are intersected at exactly 1 point, which corresponds to  $\alpha \smile \beta = [T^2]$  (be careful with the orientation). You may also try to think about how to compute the cup product of  $\Sigma_g$  in the geometric intersection way.

Another application of this geometric construction allow you compute the ring structure of  $H^*(\mathbb{C}P^n)$  differently. We set  $x_k = PD^{-1}[\mathbb{C}P^{n-k}]$ , where  $\mathbb{C}P^{n-k} \subset \mathbb{C}P^n$  as the linear subspace, as the generator  $H^{2k}(\mathbb{C}P^n)$ . Then notice that a generic  $n - i$  dimension ( $\mathbb{C}$ -linear) subspace intersection with a generic  $n - j$  dimension subspace intersection at a  $n - i - j$  dimension subspace (empty if  $i + j > n$ ), it tells you that  $x_i \smile x_j = x_{i+j}$ , and then we know  $H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1})$  again by setting  $x := x_2$ .

Nowadays, the intersection product is developed into intersection theory, which can be define in more setting (for example scheme or stack).

Finally, we present the Lefschetz fixed point theorem. (One of) its proof deeply relies on the Poincaré duality.

**Definition 8.44.** Let  $f : X \rightarrow X$  be a map and assume that  $H^q(X)$  are all finitely generated, then we define the Lefschetz number to be

$$\Lambda(X, f) := \sum_q (-1)^q \text{Tr}[f^q : H^q(X; \mathbb{Q}) \rightarrow H^q(X, \mathbb{Q})].$$

**Exercise 8.45.** (1) Show that  $\Lambda(X, \text{id}) = \chi(X)$  when it can be defined.

(2) For  $f : S^n \rightarrow S^n$ , show that

$$\Lambda(X, f) = 1 + (-1)^n \text{deg}(f).$$

**Theorem 8.46.** Let  $M$  be a compact oriented topological manifold, and  $f : M \rightarrow M$  a map. We have that if  $\Lambda(X, f) \neq 0$ , then  $f$  has a fixed point.

**Exercise 8.47.** Let  $f : \mathbb{C}P^n \rightarrow \mathbb{C}P^n$ . Here we want to study existence of fixed points of  $f$  using the cohomology ring  $H^*(\mathbb{C}P^n, \mathbb{Q}) \cong \mathbb{Q}[x]/(x^{n+1})$  and the Lefschetz fixed point theorem.

Show that: 1) If  $n$  is even, then  $f$  has a fixed point. 2) If  $n$  is odd, and  $f^*(x) \neq -x$ , then  $f$  has a fixed point.

*Supplement material 8.48.* You can also develop the Lefschetz fixed point theorem in other cohomology theories. For example, the Lefschetz fixed point theorem for  $\ell$ -adic étale cohomology plays a very basic role in solving the Weil conjecture.

## 9. HOMOLOGY FOR PAIRS AND EXCISION PRINCIPLE

In this section, we explain homology for pairs and the excision principle. This toolkit is more or less equivalent to Mayer-Vietoris, but has certain conveniences.

In this course, we will not essentially use any tools here, so you may safely skip this section. But you may need it at some moment in your life.

**9.1. Relative homology for pair and LSE.** Recall that a space pair  $(X, A)$  consists of a space  $X$  and its subspace  $A$ . To a space pair where  $i : A \subset X$  is the inclusion, we have defined

$$i_{\#} : S_*(A) \rightarrow S_*(X).$$

It is clear that this is a degree-wise injective chain map, and we can regard  $S_*(A)$  as a sub-chain complex of  $S_*(X)$  (i.e., degree-wise subgroups, and inclusion commutes with the differential since  $i_{\#}$  is a chain map).

**Definition 9.1.** We define the singular chain for the pair  $(X, A)$  (or relative chain) to be

$$S_*(X, A) := S_*(X)/S_*(A).$$

Precisely,  $S_q(X, A) = S_q(X)/S_q(A)$  and  $\partial_{S_*(X, A)}$  is descended from  $\partial_{S_*(X)}$ .

We define  $H_q(X, A)$  as  $H_q(S_*(X, A))$  as the singular homology for a pair, or relative homology.

In particular, when  $A = \emptyset$ ,  $S_q(X, \emptyset) = S_q(X)$  and  $H_q(X, A)$  as  $H_q(X)$  that recover the absolute case.

Similarly, we may also define the reduced version of cochain  $\tilde{S}^*(X; M) = \text{Hom}_{\mathbb{Z}}(\tilde{S}^*(X), M)$ , and relative version  $\tilde{S}^*(X, A; M) = \text{Hom}_{\mathbb{Z}}(\tilde{S}^*(X, A), M)$  and their cohomology  $\tilde{H}^q(X; M)$ ,  $H^q(X, A; M)$ .

*Remark 9.2.* We can develop the  $H_q(X, A; M)$  as well. To simply notation, we will not do it here.

*Remark 9.3.* In fact,  $S_q(X, A) = S_q(X)/S_q(A)$  is a free abelian group generated by singular simplexes  $\sigma : \Delta^q \rightarrow X$  with  $\text{im}(\sigma)$  does not entirely in  $A$ . But we will keep our convention to treat it as a quotient.

**Exercise 9.4.** If  $f : (X, A) \rightarrow (Y, B)$  is a map between pairs, try to define the induced map

$$f_{\#} = S_*(f) : S_*(X, A) \rightarrow S_*(Y, B), \quad f_q : H_q(X, A) \rightarrow H_q(Y, B).$$

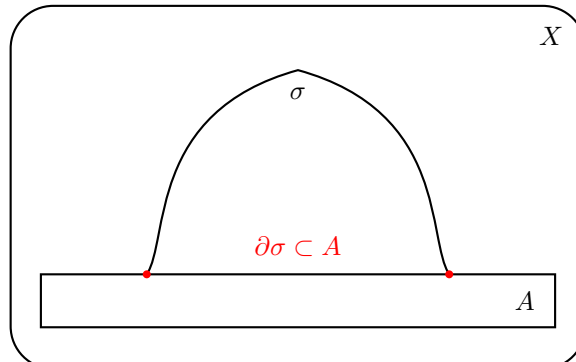
Show that  $f_*$  has homotopy invariance with respect to the relative homotopy relation. Hint: Check carefully that the homotopy invariance relation descends to relative chains.

**Exercise 9.5.** For a pair  $(X, A)$ , we consider the following construction:

$$\begin{aligned} Z'_q(X, A) &:= \{c \in S_q(X) \mid \partial_q c \in S_{q-1}(A)\}, \\ B'_q(X, A) &:= \text{im}(\partial_{q+1}) + S_q(A) \subset S_q(X). \end{aligned}$$

Show that  $B'_q(X, A) \subset Z'_q(X, A)$ , and  $H_q(X, A) \cong Z'_q(X, A)/B'_q(X, A)$ .

The construction gives a more geometric intuition for thinking of homology classes of  $H_q(X, A)$  as “cycles relative to  $A$ ”.



**Theorem 9.6.** For any space pair  $(X, A)$ , we have a long exact sequence

$$\cdots \rightarrow H_q(A) \xrightarrow{i_q} H_q(X) \xrightarrow{p_q} H_q(X, A) \xrightarrow{\partial_q} H_{q-1}(A) \rightarrow \cdots$$

If  $f : (X, A) \rightarrow (Y, B)$  is a map between pair, then  $f_q$  and  $(f|_A)_q$  are natural with respect to the long exact sequence in the sense that the following diagram is commutative

$$\begin{array}{ccccccccc} \longrightarrow & H_q(A) & \longrightarrow & H_q(X) & \longrightarrow & H_q(X, A) & \xrightarrow{\partial_q} & H_{q-1}(A) & \longrightarrow \\ & \downarrow (f|_A)_* & & \downarrow f_* & & \downarrow f_* & & \downarrow (f|_A)_* & \\ \longrightarrow & H_q(B) & \longrightarrow & H_q(Y) & \longrightarrow & H_q(X, B) & \xrightarrow{\partial_q} & H_{q-1}(B) & \longrightarrow \end{array}$$

*Proof.* By taking a degree-wise quotient, we have a degree-wise short exact sequence, which is moreover with respect to the differential

$$0 \rightarrow S_*(A) \xrightarrow{i_{\#}} S_*(X) \xrightarrow{p} S_*(X, A) \rightarrow 0.$$

The short exact sequence induces the long exact sequence by standard homological algebra. The naturality also follows since  $f_{\#}$  induces chain maps between the short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_*(A) & \longrightarrow & S_*(X) & \longrightarrow & S_*(X, A) & \longrightarrow & 0 \\ & & \downarrow (f|_A)_{\#} & & \downarrow f_{\#} & & \downarrow f_{\#} & & \\ 0 & \longrightarrow & S_*(B) & \longrightarrow & S_*(Y) & \longrightarrow & S_*(Y, B) & \longrightarrow & 0 \end{array} \quad \square$$

*Remark 9.7.* Here, let us explain a little more about the connecting map

$$H_q(X, A) \xrightarrow{\partial_q} H_{q-1}(A)$$

is given by

$$\partial_q([\sigma]) = [i_{\#}^{-1} \partial_{q, X} p^{-1} \sigma],$$

which is well defined by the argument for the snake lemma. We may compute the connecting map using the exercise below.

**Exercise 9.8.** Under the identification of Exercise 9.5, show that the connecting map  $\partial : H_q(X, A) \rightarrow H_{q-1}(A)$  can be computed as follow: For any  $z \in H_q(X, A)$ , we can find  $\sigma \in Z'_q(X, A)$  such that  $[\sigma] = z \in Z'_q(X, A)/B'_q(X, A) \cong H_q(X, A)$ , and then we have  $\partial_q(z) = [\partial_q \sigma] \in H_{q-1}(A)$  (notice that  $\sigma \in Z'_q(X, A)$  means that  $\partial_q \sigma \in Z_{q-1}(A) \subset S_{q-1}$ ).

**Exercise 9.9.** Here, we recall subsection 2.4.2. Let  $x \in X$  be a point, show that 1)  $H_q(X, x) \simeq \tilde{H}_q(X)$  for all  $q$ . 2)  $\tilde{H}_q$  also has the long exact sequence for pairs (recall the remark below).

*Remark 9.10.* You may also define  $\tilde{H}_q(X, A) := H_q(S_*(X, a)/S_*(A, a))$  for  $a \in A \subset X$ . However, you do not get anything new since  $S_*(X, a)/S_*(A, a) \cong S_*(X, A)$  as a chain complex. However, it may cause some convenience for some discussions.

**Exercise 9.11.** Let  $f : (X, A) \rightarrow (Y, B)$  is a map between pairs such that  $f : X \rightarrow Y$  and  $f|_A : A \rightarrow B$  are homotopy equivalences. Then  $f_q : H_q(X, A) \rightarrow H_q(Y, B)$  are isomorphisms. Hint: Use the five lemma and the long exact sequence.

Notice that here we do not have a relative homotopy equivalence for pairs. So, this does not follow directly from the homotopy invariance. In practice, it may simplify some discussion (for example, it may be tricky to really find a relative homotopy equivalence, but easier to check two absolute homotopy equivalences).

**Exercise 9.12.** Write down and prove the Mayer-Vietoris sequence for relative cohomology. Notice that you may also write an MV sequence where you take intersection and union in the place of  $A$  for a pair  $(X, A)$ !

Lastly, we mention that we can study a space triple  $(X, A, B)$  where  $B \subset A \subset X$ , and related long exact sequences. It may give some interesting applications.

**Exercise 9.13.** Then show that we have the following long exact sequence

$$\cdots \rightarrow H_q(A, B) \xrightarrow{i_q} H_q(X, B) \xrightarrow{j_q} H_q(X, A) \xrightarrow{\partial_q} H_{q-1}(A, B) \rightarrow \cdots,$$

where  $i : (A, B) \rightarrow (X, B)$  and  $j : (X, B) \rightarrow (X, A)$ .

For a morphism of triple  $f : (X, A, B) \rightarrow (Y, C, D)$ , we have the following commutative diagram of long exact sequences:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_q(A, B) & \longrightarrow & H_q(X, B) & \longrightarrow & H_q(X, A) & \longrightarrow & \cdots \\ & & \downarrow f_q & & \downarrow f_q & & \downarrow f_q & & \\ \cdots & \longrightarrow & H_q(C, D) & \longrightarrow & H_q(Y, D) & \longrightarrow & H_q(Y, C) & \longrightarrow & \cdots \end{array}$$

Hint: Simply repeat the proof of Theorem 9.6 for suitable chain complexes.

**9.2. Excision principle.** Now, we state the excision principle and prove it using the small chain theorem Theorem 4.2.

**Theorem 9.14** (Excision principle). *Let  $(X, A)$  be a space pair and a subspace  $U \subset A$  satisfying  $\bar{U} \subset \text{Int}(A)$ . Then the morphism of pairs*

$$f : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$$

*induces a quasi-isomorphism, i.e. isomorphisms on homologies*

$$f_q : H_q(X \setminus U, A \setminus U) \xrightarrow{\cong} H_q(X, A).$$

*Proof.* Take  $\mathcal{U} = \{X \setminus U, A\}$ . Then the condition  $\bar{U} \subset \text{Int}(A)$  implies that

$$\text{Int}(X \setminus U) \cup \text{Int}(A) = (X \setminus \bar{U}) \cup \text{Int}(A) = X.$$

Then we can use Theorem 4.2 to  $\mathcal{U}$ . Here, we take  $i_*, r_*$

$$i_* : S_*^{\mathcal{U}}(X) \subset S_*(X), \quad r_* : S_*(X) \rightarrow S_*^{\mathcal{U}}(X)$$

constructed from the small chain theorem. We also notice that in this case, we have

$$S_*^{\mathcal{U}}(X) = S_*(X \setminus U) + S_*(A).$$

Notice that  $A \in \mathcal{U}$ , so  $S_*(A)$  is a sub-chain complex of  $S_*^{\mathcal{U}}(X)$  as well as  $S_*(X)$ . The condition  $r_* i_* = \text{id}$  guarantee that  $r_*|_{S_*(A)} = \text{id}_{S_*(A)}$ . Consequently,  $r_*$  and  $i_*$  descend to chain maps

$$S_*^{\mathcal{U}}(X)/S_*(A) = [S_*(X \setminus U) + S_*(A)]/S_*(A) \hookrightarrow S_*(X)/S_*(A) = S_*(X, A).$$

Moreover,  $i_* r_* \simeq \text{id}$  through  $F_*$ , and moreover  $F_*(S_*(U_i)) \subset S_{*+1}(U_i)$  for all  $U_i \in \mathcal{U}$ . Then we have  $F_*$  also descends to a chain homotopy.

Therefore, we have  $i_*$  induces

$$H_q([S_*(X \setminus U) + S_*(A)]/S_*(A)) \cong H_q(X, A).$$

On the other hand, by the third isomorphism theorem, we have

$$S_*(X \setminus U, A \setminus U) = S_*(X \setminus U)/S_*(A \setminus U) = S_*(X \setminus U)/[S_*(X \setminus U) \cap S_*(A)] \cong [S_*(X \setminus U) + S_*(A)]/S_*(A).$$

Therefore, we have

$$H_q(X \setminus U, A \setminus U) \cong H_q([S_*(X \setminus U) + S_*(A)]/S_*(A)) \cong H_q(X, A).$$

Lastly, it remains to verify that the isomorphism is induced by  $f_q$ . We left it as an exercise.  $\square$

*Remark 9.15.* Here, we use the small chain theorem to give a direct proof of the excision principle. You may also prove the excision principle using the Mayer-Vietoris sequence for pairs. In fact, the excision principle and the Mayer-Vietoris sequence are equivalent. But the proof for the equivalence is tricky, and we left you to explore.

To apply it more effectively, we introduce the following notion.

**Definition 9.16.** We say a pair  $(X, A)$  is good, if there exists  $A \subset X$  is a closed subspace and  $A$  has an open neighborhood  $V \subset X$ , and  $A \subset V$  is a deformation retraction.

**Proposition 9.17.** *For a good pair  $(X, A)$ , we have the quotient map  $\pi : (X, A) \rightarrow (X/A, A/A)$  induces isomorphism*

$$\pi_q : H_q(X, A) \simeq H_q(X/A, A/A) = \tilde{H}_q(X/A).$$

*Proof.* We consider the following commutative diagram induced by evident maps between pairs

$$\begin{array}{ccccc} H_q(X, A) & \xrightarrow{a} & H_q(X, V) & \xleftarrow{b} & H_q(X \setminus A, V \setminus A) \\ \downarrow \pi_q & & \downarrow \pi_q & & \downarrow \pi_q \\ H_q(X/A, A/A) & \xrightarrow{c} & H_q(X/A, V/A) & \xleftarrow{d} & H_q(X/A \setminus A/A, V/A \setminus A/A) \end{array}$$

By Exercise 9.11, we have  $a, c$  are isomorphisms. By Theorem 9.14, we have  $b, d$  are isomorphisms.

However, the right most  $\pi_q$  is an isomorphism since  $\pi$  restrict to a homeomorphism between  $(X \setminus A, V \setminus A)$  and  $(X/A \setminus A/A, V/A \setminus A/A)$  by definition of quotient space.

Then the left most  $\pi_q$  is an isomorphism by the commutativity of the diagram.  $\square$

**Exercise 9.18.** For  $n \geq 1$ , we have  $H_q(D^n, S^{n-1}) = \mathbb{Z}$  for  $q = n$ , and otherwise trivial. Hint: Use Proposition 9.17 and the computation of  $\tilde{H}_q(S^n)$ .

**Exercise 9.19.** Let  $f : X \rightarrow Y$  be a map. Consider the mapping cone  $C(f)$ , see Example 4.26. We can treat  $Y$  as a subspace of  $C(f)$  via the map  $Y \rightarrow Y \sqcup C(X) \rightarrow C(f) = (Y \sqcup C(X))/\sim$  (check this is a closed embedding), then show that  $(C(f), Y)$  is a good pair and  $C(f)/Y \cong \Sigma X$ . Consequently, we have  $\tilde{H}_q(\Sigma X) \cong H_q(C(f), Y)$ , and you may prove the mapping cone sequence using the long exact sequence for pairs. One particular case is the closed inclusion  $i : A \subset X$ , then we have  $H_q(X, A) = \tilde{H}_q(C(i))$ .

**Exercise 9.20.** For a CW complex  $X$ , show that  $(X^q, X^{q-1})$  is a good pair for all  $q \geq 1$ . Then we have  $C_q(X) \cong H_q(X^q, X^{q-1})$ . This is another way to define cellular homology (try to use the previous exercise to show two definitions are equivalent).

*Supplement material 9.21.* We already see that homology for mapping cone and quotient are deeply related. This is a pattern of a more general notion of cofibration (see [Die08, Chapter 5]), and we use the special case:  $A \rightarrow X$  is a cofibration when  $(X, A)$  is a good pair. The notion is very useful in further study of homotopy theory.

### 9.3. Relative cohomology.

**Definition 9.22.** We define the reduced version of cochain  $\tilde{S}^*(X; M) = \text{Hom}_{\mathbb{Z}}(\tilde{S}^*(X), M)$ , and relative version  $\tilde{S}^*(X, A; M) = \text{Hom}_{\mathbb{Z}}(\tilde{S}^*(X, A), M)$  and their cohomology  $\tilde{H}^q(X; M)$ ,  $H^q(X, A; M)$ .

*Remark 9.23.* To relative cochain  $c \in S^q(X, A; M)$ ,

$$c : S_q(X)/S_q(A) \rightarrow M,$$

we can literally think it as a  $c \in S^q(X; M)$  such that  $c|_{S_q(A)} = 0$ . The reason is the following: for chains, we have  $S_q(X) \rightarrow S_q(X)/S_q(A)$  is a surjective homomorphism, but after applying  $\text{Hom}_{\mathbb{Z}}(-, M)$ , we have an injective homomorphism (see Exercise 7.1)

$$S^q(X, A; M) \hookrightarrow S^q(X; M),$$

whose image can be characterized by  $c \in S^q(X; M)$  such that  $c|_{S_q(A)} = 0$ . And in the case of cochain, we have  $S^q(X) \rightarrow S^q(A)$  is a quotient (looks a little weird, but this is the feature! Not a bug.)

**Theorem 9.24.** *For a map  $f : (X, A) \rightarrow (Y, B)$ , we have induced cochain map  $S^*(f) = f^\#$  and its cohomology  $H^q(f) = f^q$*

$$f^\# : S^*(Y, B) \rightarrow S^*(X, A), \quad f^q : H^q(Y, B) \rightarrow H^q(X, A),$$

which are compatible with composition.

When  $f \simeq g$ , we have  $f^\# \simeq g^\#$  (as cochain maps), and then  $f^q = g^q$ .

**Theorem 9.25.** *We have the following long exact sequences:*

(1) *For any space pair  $(X, A)$ , we have a long exact sequence*

$$\cdots \rightarrow H^q(X, A) \xrightarrow{a} H_q(X) \xrightarrow{i^q} H_q(A) \xrightarrow{\delta} H_{q+1}(X, A) \rightarrow \cdots$$

*If  $f : (X, A) \rightarrow (Y, B)$  is a map between pair, then  $f^q$  and  $(f|_A)^q$  are natural with respect to the long exact sequence in the sense that the following diagram is commutative*

$$\begin{array}{ccccccc} \longrightarrow & H^q(X, A) & \longrightarrow & H^q(X) & \longrightarrow & H^q(A) & \xrightarrow{\delta} & H^{q+1}(X, A) & \longrightarrow \\ & \downarrow f^* & & \downarrow f^* & & \downarrow (f|_A)^* & & \downarrow f^* & \\ \longrightarrow & H^q(Y, B) & \longrightarrow & H^q(Y) & \longrightarrow & H^q(B) & \xrightarrow{\delta} & H^{q+1}(Y, B) & \longrightarrow \end{array}$$

(2) *Let  $(X, A)$  be a space pair and a subspace  $U \subset A$  satisfying  $\bar{U} \subset \text{Int}(A)$ . Then the morphism of pair*

$$f : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$$

*induces an quasi-isomorphisms, i.e. isomorphisms on homologies*

$$f^q : H^q(X, A) \xrightarrow{\cong} H^q(X \setminus U, A \setminus U).$$

*Proof.* (1) Similar to Theorem 7.6, we only need that

$$0 \rightarrow S^*(X, A) \xrightarrow{p} S_*(X) \xrightarrow{i_{\#}} S^*(A) \rightarrow 0$$

is exact since  $S_*(A)$  is a degree-wise free chain complex. The chain level result

(2) Simple the dual of Theorem 9.14 because we have a chain homotopy there.  $\square$

**Corollary 9.26.** *For a good pair  $(X, A)$ , we have the quotient map  $\pi : (X, A) \rightarrow (X/A, A/A)$  induces isomorphism*

$$\pi^q : H^q(X/A, A/A) = \tilde{H}^q(X/A) \simeq H^q(X, A).$$

**9.4. Relative cup product.** Here, we mention the relative cup product. Recall that the chain-level cup product is the bilinear map (written explicitly)

$$S^p(X) \otimes S^q(X) \rightarrow S^{p+q}(X), c \otimes d \mapsto c \smile d = [\sigma \in S_{p+q}(X) \mapsto c(p\sigma)d(\sigma_q)].$$

Now, as we have Remark 9.23, we regard  $c \in S^p(X, A)$  as  $c : S_p(X) \rightarrow \mathbb{Z}$  that restricts to 0 on  $S_p(A)$ .

**Lemma 9.27.** *Let  $c \in S^p(X, A)$  and  $d \in S^q(X, B)$ , then for  $\sigma \in S_{p+q}(A) + S_{p+q}(B) \subset S_{p+q}(X)$ , we have  $c \smile d(\sigma) = c(p\sigma)d(\sigma_q) = 0$ .*

*Proof.* By definition of  $\sigma$  we have  $\sigma = \sigma_A + \sigma_B$ , where  $\sigma_A \in S_{p+q}(A)$  and  $\sigma_B \in S_{p+q}(B)$ . Without loss of generality, we assume that  $\sigma = \sigma_A$ . Then  $p(\sigma_A) \in S_p(A)$  and  $(\sigma_A)_q \in S_q(A)$ .  $\square$

**Proposition 9.28.** *If  $A, B$  is a Mayer-Vietoris duo, then the chain level cup product induces a relative cup product*

$$\smile : H^p(X, A) \otimes H^q(X, B) \rightarrow H^{p+q}(X, A \cup B).$$

*Proof.* The previous lemma shows that the chain level cup products a bilinear map

$$\smile : S^p(X, A) \otimes S^q(X, B) \rightarrow \text{Hom}_{\mathbb{Z}}(S_{p+q}(X)/[S_{p+q}(A) + S_{p+q}(B)], \mathbb{Z}), c \otimes d \mapsto c \smile d.$$

Moreover,  $\{A, B\}$  is a Mayer-Vietoris duo, then we have the natural map

$$S_{p+q}(X)/[S_{p+q}(A) + S_{p+q}(B)] \rightarrow S_{p+q}(X)/[S_{p+q}(A \cup B)]$$

induces a quasi-isomorphism

$$\text{Hom}_{\mathbb{Z}}(S_*(X)/[S_*(A \cup B)], \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(S_*(X)/[S_*(A) + S_*(B)], \mathbb{Z})$$

by the 5-lemma. Then all the rest things are routine linear algebra.  $\square$

*Remark 9.29.* We often use the following cases,  $\{A, B\} = \{A\}$  or  $\{A, B\} = \{A, \emptyset\}$ , where the condition of the proposition is automatically true.

Similarly, you can define relative cap product and relative cross product.

Those relative products can be used to state and prove some relative version of Poincaré duality, which is useful in the study of vector bundles. For example, the Thom isomorphism theorem and Gysin sequence follow from that, and they are also related to the characteristic classes theory. We refer to [Die08, Chapter 17, 18] for more details.

## APPENDIX A. CHEAT SHEET ON HOMOLOGICAL ALGEBRA

Here, we should recall exactness and useful situations. Also, the five lemma and the snake lemma. We refer to [Wei94, Chapter 1-3] for more details.

In the following, we state in the homological convention (i.e., the differential decrease degree by 1), you may easily write down the same result for the cohomological convention.

## A.1. Exactness.

**Definition A.1.** Let

$$\cdots \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} \cdots$$

be a sequence of abelian groups and group homomorphisms. We say the sequence is *exact at*  $A_i$  if

$$\text{im}(f_{i-1}) = \ker(f_i).$$

A sequence is called *exact* if it is exact at every term.

The following exact sequence of 5-terms is called a short exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

Equivalently,  $f$  is injective,  $g$  is surjective, and  $\text{im}(f) = \ker(g)$ .

We often split long exact sequences into many short exact sequences.

**Lemma A.2.** Let

$$\cdots \rightarrow A_{q+1} \xrightarrow{f_{q+1}} A_q \xrightarrow{f_q} A_{q-1} \rightarrow \cdots$$

be a long exact sequence of abelian groups. Then for every  $q$  there is a natural short exact sequence

$$0 \rightarrow \text{coker}(f_{q+1}) \rightarrow A_q \rightarrow \ker(f_{q-1}) \rightarrow 0.$$

**Definition A.3.** A short exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

is said to *split* if one of the following equivalent conditions holds:

- (1) there exists a homomorphism  $s : C \rightarrow B$  such that  $g \circ s = \text{id}_C$ ;
- (2) there exists a homomorphism  $r : B \rightarrow A$  such that  $r \circ f = \text{id}_A$ ;
- (3)  $B \cong A \oplus C$  and under this identification  $f$  and  $g$  are the standard inclusion and projection.

**Proposition A.4.** If

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a short exact sequence and  $C$  is a free abelian group, then the sequence splits.

**Proposition A.5** (Five lemma). Consider a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5. \end{array}$$

If  $f_1, f_2, f_4, f_5$  are isomorphisms, then  $f_3$  is an isomorphism.

**Theorem A.6** (Snake lemma). Consider a commutative diagram of abelian groups

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \xrightarrow{g} & C' \end{array}$$

whose rows are exact. Then there exists a natural exact sequence

$$\ker(x) \rightarrow \ker(\alpha) \rightarrow \ker(\beta) \rightarrow \ker(\gamma) \xrightarrow{\partial} \text{coker}(\alpha) \rightarrow \text{coker}(\beta) \rightarrow \text{coker}(\gamma) \rightarrow \text{coker}(y).$$

The homomorphism  $\partial : \ker(\gamma) \rightarrow \text{coker}(\alpha)$  is called the connecting morphism.

## A.2. Chain complexes.

**Definition A.7.** A chain complex  $(C_*, \partial_*)$  of abelian groups consists of a sequence of abelian groups and homomorphisms

$$\cdots \xrightarrow{\partial_{q+2}} C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \xrightarrow{\partial_{q-1}} \cdots$$

such that

$$\partial_q \circ \partial_{q+1} = 0$$

for every  $q$ .

**Definition A.8.** Let  $(C_*, \partial_*^C)$  and  $(D_*, \partial_*^D)$  be chain complexes. A chain map  $f_* : C_* \rightarrow D_*$  is a collection of homomorphisms

$$f_q : C_q \rightarrow D_q$$

for all  $q$ , such that for every  $q$  the following diagram commutes:

$$\begin{array}{ccc} C_q & \xrightarrow{\partial_q^C} & C_{q-1} \\ \downarrow f_q & & \downarrow f_{q-1} \\ D_q & \xrightarrow{\partial_q^D} & D_{q-1} \end{array}$$

that is,

$$f_{q-1} \circ \partial_q^C = \partial_q^D \circ f_q$$

for every  $q$ .

**Proposition A.9.** Let  $f : C_* \rightarrow D_*$  be a chain map between chain complexes. Then  $f$  induces homomorphisms

$$H_q(f) : H_q(C_*) \rightarrow H_q(D_*)$$

for all  $q$ . Moreover, if  $g : D_* \rightarrow E_*$  is another chain map, then

$$H_q(g \circ f) = H_q(g) \circ H_q(f), \quad H_q(\text{id}_{C_*}) = \text{id}_{H_q(C_*)}.$$

**Proposition A.10.** Let

$$0 \rightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \rightarrow 0$$

be a short exact sequence of chain complexes. Then there is a natural long exact sequence on homology

$$\cdots \rightarrow H_q(A_*) \xrightarrow{H_q(f)} H_q(B_*) \xrightarrow{H_q(g)} H_q(C_*) \xrightarrow{\partial} H_{q-1}(A_*) \rightarrow \cdots$$

**Definition A.11.** Let  $f, g : C_* \rightarrow D_*$  be chain maps. A chain homotopy from  $f$  to  $g$  is a family of homomorphisms

$$h_q : C_q \rightarrow D_{q+1}$$

such that

$$\partial h + h \partial = f - g.$$

In this case we write  $f \simeq g$ .

**Proposition A.12.** If two chain maps  $f, g : C_* \rightarrow D_*$  are chain homotopic, then they induce the same maps on homology:

$$H_q(f) = H_q(g)$$

for all  $q$ .

**Definition A.13.** A chain map  $f : C_* \rightarrow D_*$  is called a quasi-isomorphism if

$$H_q(f) : H_q(C_*) \rightarrow H_q(D_*)$$

is an isomorphism for every  $q$ .

**Definition A.14.** Two chain complexes  $C_*$  and  $D_*$  are *chain homotopy equivalent* if there exist chain maps

$$f : C_* \rightarrow D_*, \quad g : D_* \rightarrow C_*$$

such that

$$g \circ f \simeq \text{id}_{C_*}, \quad f \circ g \simeq \text{id}_{D_*}.$$

**Proposition A.15.** *If two chain complexes are chain homotopy equivalent, then they are quasi-isomorphic.*

**Proposition A.16.** *Let*

$$C_* = \bigoplus_{\alpha \in I} C_*^{(\alpha)}$$

*be a direct sum of chain complexes. Then*

$$H_q(C_*) \cong \bigoplus_{\alpha \in I} H_q(C_*^{(\alpha)})$$

*for every  $q$ .*

**Theorem A.17** (Algebraic universal coefficient theorem for homology). *Let  $C_*$  be a chain complex of free abelian groups, and let  $M$  be an abelian group. Then for every  $q$  there is a natural short exact sequence*

$$0 \rightarrow H_q(C_*) \otimes_{\mathbb{Z}} M \rightarrow H_q(C_* \otimes_{\mathbb{Z}} M) \rightarrow \text{Tor}_1^{\mathbb{Z}}(H_{q-1}(C_*), M) \rightarrow 0.$$

*Moreover, this short exact sequence splits, though not naturally in general.*

**Theorem A.18** (Algebraic universal coefficient theorem for cohomology). *Let  $C_*$  be a chain complex of free abelian groups, and let  $M$  be an abelian group. Then for every  $q$  there is a natural short exact sequence*

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{q-1}(C_*), M) \rightarrow H^q(\text{Hom}_{\mathbb{Z}}(C_*, M)) \rightarrow \text{Hom}_{\mathbb{Z}}(H_q(C_*), M) \rightarrow 0.$$

*Moreover, this short exact sequence splits, though not naturally in general.*

**Theorem A.19** (Algebraic Künneth theorem). *Let  $C_*$  and  $D_*$  be chain complexes of abelian groups. Assume that one of  $C_*$  and  $D_*$  is a chain complex of free abelian groups. Then for every  $n$  there is a natural short exact sequence*

$$0 \rightarrow \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(D_*) \rightarrow H_n(C_* \otimes D_*) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}_1^{\mathbb{Z}}(H_p(C_*), H_q(D_*)) \rightarrow 0.$$

*Moreover, this short exact sequence splits, though not naturally in general.*

**Remark A.20.** Those 3 theorems are about derived tensor / derived Hom. So, freeness is sufficient to compute the derived tensor (free implies flat) / derived Hom (free implies projective).

Let  $A, B, M$  be abelian groups. The following identities are frequently used.

(1)  $\text{Tor}_1^{\mathbb{Z}}(A, B) \cong \text{Tor}_1^{\mathbb{Z}}(B, A)$ .

(2)  $\text{Ext}_{\mathbb{Z}}^1(A, B)$  is contravariant in  $A$  and covariant in  $B$ , while  $\text{Tor}_1^{\mathbb{Z}}(A, B)$  is covariant in both variables.

(3) If  $F$  is a flat abelian group (for example, free or fields of characteristic 0),  $P$  is a projective abelian group (for example, free), and  $I$  is an injective abelian group (for example,  $I = \mathbb{Q}/\mathbb{Z}$ ), then

$$\text{Tor}_1^{\mathbb{Z}}(F, M) = 0, \quad \text{Ext}_{\mathbb{Z}}^1(P, M) = 0, \quad \text{Ext}_{\mathbb{Z}}^1(A, I) = 0.$$

(4) For every  $n \geq 1$ ,

$$M \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong M/nM, \quad \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, M) \cong \{m \in M \mid nm = 0\}, \quad \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n, M) \cong M/nM.$$

(5) For every  $m, n \geq 1$ ,

$$\begin{aligned} \mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n &\cong \mathbb{Z}/\text{gcd}(m, n), \\ \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) &\cong \mathbb{Z}/\text{gcd}(m, n), \\ \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m, \mathbb{Z}/n) &\cong \mathbb{Z}/\text{gcd}(m, n). \end{aligned}$$

(6) We have

$$\mathrm{Tor}_1^{\mathbb{Z}}\left(\bigoplus_i A_i, B\right) \cong \bigoplus_i \mathrm{Tor}_1^{\mathbb{Z}}(A_i, B), \quad \mathrm{Tor}_1^{\mathbb{Z}}\left(A, \bigoplus_j B_j\right) \cong \bigoplus_j \mathrm{Tor}_1^{\mathbb{Z}}(A, B_j);$$

and

$$\mathrm{Ext}_{\mathbb{Z}}^1\left(\bigoplus_i A_i, M\right) \cong \prod_i \mathrm{Ext}_{\mathbb{Z}}^1(A_i, M), \quad \mathrm{Ext}_{\mathbb{Z}}^1\left(A, \bigoplus_j B_j\right) \cong \bigoplus_j \mathrm{Ext}_{\mathbb{Z}}^1(A, B_j)$$

whenever  $A$  is finitely generated.

(7) If  $A \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^k \mathbb{Z}/n_i$ , then

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, M) \cong \bigoplus_{i=1}^k \{m \in M \mid n_i m = 0\},$$

$$\mathrm{Ext}_{\mathbb{Z}}^1(A, M) \cong \bigoplus_{i=1}^k M/n_i M.$$

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