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Capacities from the Chiu-Tamarkin complex

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To Claude Viterbo on the Occasion of his Sixtieth Birthday

In this paper, we construct a sequence $(c_k)_{k\in\mathbb{N}}$ of symplectic capacities based on the Chiu-Tamarkin complex $C_T^{\mathbb{Z}/\ell}$, a $\mathbb{Z}/\ell$-equivariant invariant coming from the microlocal theory of sheaves. We compute $(c_k)_{k\in\mathbb{N}}$ for convex toric domains, which are the same as the Gutt-Hutchings capacities. Our method also works for the prequantized contact manifold $T^*X \times S^1$. We define a sequence of "contact capacities" $([c]_k)_{k\in\mathbb{N}}$ on the prequantized contact manifold $T^*X \times S^1$, and we compute them for prequantized convex toric domains.

0. Introduction

0.1. Symplectic embedding

A symplectic manifold (X, ω) is a manifold with a non-degenerate closed 2-form ω. Classically it appears naturally as phase spaces in Hamiltonian Mechanics. An embedding $\varphi : (X, \omega) \hookrightarrow (X', \omega')$ is called symplectic if $\varphi^*\omega' = \omega$. A basic question in symplectic geometry is to decide when a symplectic embedding between two symplectic manifolds exists. The first result, the origin of the question, is the Gromov non-squeezing theorem:

Theorem 0.A ([Gro85]). *Equip $\mathbb{R}^{2d}$ with the linear symplectic form. Let $B_{\pi r^2} = \{(x,p) \in \mathbb{R}^{2d} : \|x\|^2 + \|p\|^2 < r^2\}$, and $Z_{\pi R^2} = \{(x,p) \in \mathbb{R}^{2d} : x_1^2 + p_1^2 < R^2\}$.*

If there is a symplectic embedding $\varphi : B_{\pi r^2} \to \mathbb{R}^{2d}$ such that $\varphi(B_{\pi r^2}) \subset Z_{\pi R^2}$, then $r \leq R$.

A structure related to the embedding question is the so-called symplectic capacity. One example is the Gromov width, which is hard to compute. There are other capacities defined by generating functions [Vit92], Hamiltonian dynamics [EH90], and J-holomorphic curves [Hut11, GH18, Sie19]. A great survey about symplectic capacities is [CHLS10]. When the dimension is 4, the ECH capacity is a very effective tool. When the dimension is greater than 4, we know fewer results.

The Ekeland-Hofer capacity $(c_k^{\mathrm{EH}})_{k\in\mathbb{N}}$ is a sequence of symplectic capacities defined for compact star-shaped domains in $T^*\mathbb{R}^d$ for all d, which is defined using Hamiltonian dynamics. The computation of c_k^{EH} is known for ellipsoids and poly-disks, say:

$$c_k^{\mathrm{EH}}(E(a_1,\dots,a_d)) = \min\left\{T : \sum_{i=1}^{d} \lfloor \frac{T}{a_i} \rfloor \geq k\right\} \quad c_k^{\mathrm{EH}}(D(a_1,\dots,a_d)) = ka_1,$$

where

$$\begin{aligned} E(a_1,\dots,a_d) &= \left\{u \in \mathbb{C}^d : \sum_{i=1}^{d} \frac{\pi|u_i|^2}{a_i} < 1\right\}, \\ D(a_1,\dots,a_d) &= \left\{u \in \mathbb{C}^d : \pi|u_i|^2 < a_i, \forall i = 1,\dots,d\right\}, \\ &\text{with } 0 < a_1 \leq \cdots \leq a_d. \end{aligned}$$

On the other hand, Gutt and Hutchings constructed a sequence of capacities $(c_k^{\mathrm{GH}})_{k\in\mathbb{N}}$ in [GH18] using the positive S^1-equivariant symplectic homology. For an open set $\Omega \subset \mathbb{R}^d$, the open set

$$X_\Omega = \left\{u \in \mathbb{C}^d : (\pi|u_1|^2,\dots,\pi|u_d|^2) \in \Omega\right\}$$

is called a toric domain. We say X_Ω is convex if $\widehat{\Omega} = \{(x_1,\dots,x_d) : (|x_1|,\dots,|x_d|) \in \Omega\}$ is convex, and is concave if $\mathbb{R}^d_{\geq 0} \setminus \Omega$ is convex. The toric domain X_Ω is determined by $\Omega \cap \mathbb{R}^d_{\geq 0}$. So it is free to choose a suitable Ω. In particular, we always assume $\mathbb{R}^d_{\leq 0} \subset \Omega$. If X_Ω is a convex or a concave toric domain, one can indeed take Ω to be convex or concave (in the usual sense) and satisfying the condition $\mathbb{R}^d_{\leq 0} \subset \Omega$. For example, ellipsoids $E(a) = X_{\Omega_{E(a)}}$ and poly-disks $D(a) = X_{\Omega_{D(a)}}$ are convex toric domains, where

$$\Omega_{E(a)} = \left\{ (x_1, \dots, x_d) : \sum_{i=1}^d \frac{x_i}{a_i} < 1 \right\},$$
$$\Omega_{D(a)} = \left\{ (x_1, \dots, x_d) : x_i < a_i, \, \forall i = 1, \dots, d \right\}.$$

Gutt and Hutchings computed c_k^{GH} for both convex and concave toric domains. For example, when X_Ω is convex, they showed that

$$\begin{aligned} c_k^{\mathrm{GH}}(X_\Omega) &= \min \left\{ \|v\|_\Omega^* : v \in \mathbb{N}^d, \sum_{i=1}^d v_i = k \right\} \\ &= \inf \left\{ T \geq 0 : \exists z \in \Omega_T^\circ, I(z) \geq k \right\}, \end{aligned} \tag{0.1}$$

where

$$\begin{aligned} &\|v\|_\Omega^* = \max\{\langle v, w\rangle : w \in \Omega\}, \\ &\Omega_T^\circ = \{z \in \mathbb{R}^d : T + \langle z, \zeta\rangle \geq 0, \forall \zeta \in \Omega\}, \quad I(z) = \sum_{i=1}^d \lfloor -z_i \rfloor. \end{aligned} \tag{0.2}$$

So one can observe that for ellipsoids and poly-disks, $c_k^{\mathrm{GH}} = c_k^{\mathrm{EH}}$.

Unfortunately, even for ellipsoids, we know that the obstructions, given by Ekeland-Hofer capacities, and then the Gutt-Hutchings capacities, are not sharp. One new progress on higher dimensional embeddings is given by Siegel in [Sie19, Sie21]. Siegel gave sharp obstructions for embeddings between some stabilized ellipsoids.

In this paper, we construct a sequence of symplectic capacities $(c_k)_{k\in\mathbb{N}}$ for open sets in a cotangent bundle T^*X with an orientable base X. We denote $\mathrm{Open}(T^*X)$ the set of open sets in T^*X. Our main ingredient is the complex $C_T^{\mathbb{Z}/\ell}(U, \mathbb{K})$ defined by Tamarkin and Chiu in [Tam15, Chi17], where U is admissible (for example bounded open sets), $T \geq 0$, and $\ell \in \mathbb{N}_{\geq 2}$. There exists a structure of $\mathbb{K}[u]$-module on $H^*C_T^{\mathbb{Z}/\ell}(U, \mathbb{K})$, and a fundamental class $\eta_T^{\mathbb{Z}/\ell}(U, \mathbb{K}) \in H^0C_T^{\mathbb{Z}/\ell}(U, \mathbb{K})$. For admissible open sets U, we define (see

Definition 2.24)

$$\mathrm{Spec}(U,k) \coloneqq \left\{ T \geq 0 : \begin{matrix} \exists p \text{ prime such that } \forall \ell \in \mathbb{N}_{\geq 2},\ p_\ell \geq p, \\ \eta_T^{\mathbb{Z}/\ell}(U,\mathbb{F}_{p_\ell}) \in u^k H^* C_T^{\mathbb{Z}/\ell}(U,\mathbb{F}_{p_\ell}) \end{matrix} \right\},$$

and

$$c_k(U) \coloneqq \inf \mathrm{Spec}(U,k) \in [0,+\infty].$$

For a general open set U, we define $c_k(U) = \sup\{c_k(O) : O \subset U,\ O \text{ is admissible}\}$. Then we show

Theorem 0.B (Theorem 2.25). *The functions $c_k : Open(T^*X) \to [0,\infty]$ satisfy the following:*

1) $c_k \leq c_{k+1}$ for all $k \in \mathbb{N}$.

2) For two open sets $U_1 \subset U_2$, we have $c_k(U_1) \leq c_k(U_2)$.

3) For a compactly supported Hamiltonian isotopy $\varphi : I \times T^*X \to T^*X$, we have $c_k(U) = c_k(\varphi_z(U))$.

4) If $X = \mathbb{R}^d$, then $c_k(rU) = r^2 c_k(U)$ for all $k \in \mathbb{N}$ and $r > 0$.

5) Suppose $U = \{H < 1\}$ is admissible such that $\partial U = \{H = 1\}$ is a non-degenerated hypersurface of restricted contact type defined by a Hamiltonian function H. If $c_k(U) < \infty$, then $c_k(U)$ is represented by the action of a closed characteristic in the boundary ∂U.

6) $c_k(U) > 0$ for all open sets U.

Moreover, based on the structural theorem (Theorem 3.6) of $H^* C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$, where X_Ω is a convex toric domain, we can compute $c_k(X_\Omega)$ as follows:

$$c_k(X_\Omega) = \inf \left\{ T \geq 0 : \exists z \in \Omega_T^\circ, I(z) \geq k \right\}. \tag{0.3}$$

Therefore, $c_k(X_\Omega) = c_k^{\mathrm{GH}}(X_\Omega)$ by (0.1) and (0.3).

On the other hand, one may ask the concave case. It is explained in Remark 3.5 that some technical issues exist. So we cannot derive a clear structure theorem as Theorem 3.6, and then the computation of capacities is not completely clear. However manual computation of some examples shows the coincidence with Gutt-Hutchings capacities is still true.

Based on the computation on the convex toric domains and concave toric domains, Gutt and Hutchings conjectured ([GH18, Conjecture 1.9]) that, for a bounded star-shaped domain U and for all $k \in \mathbb{N}$,

$$c_k^{\mathrm{EH}}(U) = c_k^{\mathrm{GH}}(U).$$

In fact, the result $c_1^{\mathrm{EH}}(U) = c_1^{\mathrm{GH}}(U) =$ Minimal action has been proven by Irie[Iri22] for convex body U. Comparing to our result, we hope the consistency could be extended to c_k as well.

Conjecture 0.C. *For a bounded star-shaped domain $U \subset \mathbb{R}^{2d}$ and for all $k \in \mathbb{N}$, there is*

$$c_k^{\mathrm{EH}}(U) = c_k^{\mathrm{GH}}(U) = c_k(U).$$

Remark 0.D. Recently, Jean Gutt and Vinicius G. B. Ramos claim that they prove $c_k^{\mathrm{EH}}(U) = c_k^{\mathrm{GH}}(U)$ for star-shaped domain $U \subset \mathbb{R}^{2d}$.

0.2. Contact embedding

Contact geometry is the odd-dimensional cousin of symplectic geometry. A (co-oriented) contact manifold (X, α) consists of a manifold X of dimension $2n+1$ and a 1-form α such that $\alpha \wedge d\alpha^n \neq 0$. An embedding $\varphi : (X, \alpha) \hookrightarrow (X', \alpha')$ between two contact manifolds is called contact if $\varphi^*\alpha' = e^f \alpha$ for some function $f \in C^\infty(X)$. We also study the embedding question in contact geometry. The pioneering work of Eliashberg, Kim, and Polterovich[EKP06] promote our understanding of the contact embedding question a lot. Let us explain here.

A naive attempt is to study the non-squeezing problem in the 1-jet bundle $J^1\mathbb{R}^d = T^*\mathbb{R}^d \times \mathbb{R}$ equipped with the contact form $\alpha = dt + \mathbf{q}d\mathbf{p}$. However the re-scaling map $(\mathbf{q}, \mathbf{p}, t) \mapsto (r\mathbf{q}, r\mathbf{p}, r^2 t)$, which is a contactomorphism, squeezes any compact set into an arbitrary small neighborhood of the origin when r is big enough. This conformal naturality of 1-jet space illustrates us that we can study the prequantized space $T^*\mathbb{R}^d \times S^1_\sigma$, where S^1_σ is a circle. Here we equip $T^*\mathbb{R}^d \times S^1_\sigma$ with a contact form $\alpha = d\sigma + \frac{1}{2}(\mathbf{q}d\mathbf{p} - \mathbf{p}d\mathbf{q})$. However there is a global contactomorphism $F_N : T^*\mathbb{R}^d \times S^1_\sigma \to T^*\mathbb{R}^d \times S^1_\sigma$ defined as follows: We use complex coordinates $T^*\mathbb{R}^d \cong \mathbb{C}^d$, and then $F_N(z, \sigma) := (\nu(\sigma)e^{2\pi N\sigma}z, \sigma)$, where $\nu(\sigma) = (1 + N\pi|z|^2)^{-1/2}$. One can compute directly that F_N is still embedding any ball into arbitrary small neighborhood of $\{0\} \times S^1$ for N big enough. However we notice that F_N is not compactly supported. So loc. cit. proposed the following definition.

Definition. [EKP06, p1636] Let (W, α) be a contact manifold. If $U_1, U_2 \subset W$ are two open subsets, we say that U_1 is squeezed into U_2 if there exists a compactly support contact isotopy $\varphi : [0,1]_s \times \overline{U_1} \to W$ such that $\varphi_0 = \mathrm{Id}$, and $\varphi_1(\overline{U_1}) \subset U_2$.

An interesting phenomenon, which does not appear in the symplectic situation, is the scale of the ball will affect the validity of squeezing. About the large scale phenomenon, Eliashberg, Kim, and Polterovich give a very nice physical explanation using the quantization process. Two results about both squeezing and non-squeezing of prequantized balls $B_{\pi R^2} \times S^1$ are:

Theorem 0.E. *1) [EKP06, Theorem 1.3] Suppose $d \geq 2$. Then for all $0 < \pi r^2, \pi R^2 < 1$, one can squeeze the prequantized ball $B_{\pi R^2} \times S^1$ into $B_{\pi r^2} \times S^1$ whatever the relation between r and R is.*

2) [EKP06, Theorem 1.2] If there exists an integer $m \in [\pi r^2, \pi R^2]$, then $B_{\pi R^2} \times S^1$ cannot be squeezed into $B_{\pi r^2} \times S^1$.

Then the only case left about the contact non-squeezing is: what will happen if there is an integer m such that $m < \pi r^2 < \pi R^2 < m + 1$? It is solved by Chiu using the microlocal theory of sheaves[Chi17], and by Fraser using technique of J-holomorphic curves[Fra16] in the spirit of [EKP06]. They proved the following:

Theorem 0.F ([Chi17, Fra16]). *If $1 \leq \pi r^2 < \pi R^2$, then $B_{\pi R^2} \times S^1$ cannot be squeezed into $B_{\pi r^2} \times S^1$.*

The second purpose of the paper is to explain how Chiu's work could be used to define "contact capacities" on prequantization $T^*X \times S^1$ for orientable X. The notion of admissible open sets still makes sense. The presence of the scale feature makes us consider the so-called big admissible open sets. Concretely, we say a contact admissible open set $\mathscr{U} \subset T^*\mathbb{R}^d \times S^1$ is big if there is a ball $B_a \times S^1$ for $a > 1$ that can be embedded in $\mathscr{U}$ by a compactly supported contact isotopy on $T^*\mathbb{R}^d \times S^1$. For a prequantized convex toric domain $X_\Omega \times S^1$, where X_Ω is a toric domain, it is big if $\|\Omega_1^\circ\|_\infty = \max_{z \in \Omega_1^\circ} \|z\|_\infty < 1$. Besides, to obtain the contact invariance, we need to restrict to $T/\ell \in \mathbb{N}$ situation in the contact case. Here, we denote $\mathbb{P}$ the set of all prime numbers. Then we can define

$$[\mathrm{Spec}](\mathscr{U}, k) := \{p \in \mathbb{P} : \eta_p^{\mathbb{Z}/p,c}(\mathscr{U}, \mathbb{F}_p) \in u^k H^* \mathscr{C}_p^{\mathbb{Z}/p}(\mathscr{U}, \mathbb{F}_p)\}$$

and

$$[c]_k(\mathscr{U}) := \min[\mathrm{Spec}](\mathscr{U}, k) \in \mathbb{P}.$$

For a general open set $\mathscr{U}$, we define $[c]_k(\mathscr{U}) = \sup\{[c]_k(O) : O \subset \mathscr{U}, O \text{ is admissible}\}$.

Then we have, Theorem 4.9, Theorem 4.11:

Theorem 0.G. *The functions $[c]_k : Open(T^*X \times S^1) \to \mathbb{P}$ satisfy the following:*

1) $[c]_k \leq [c]_{k+1}$ for all $k \in \mathbb{N}$.

2) For two open sets $\mathscr{U}_1 \subset \mathscr{U}_2$, we have $[c]_k(\mathscr{U}_1) \leq [c]_k(\mathscr{U}_2)$.

3) For a compactly supported contact isotopy $\varphi : I \times T^*X \times S^1 \to T^*X \times S^1$, we have $[c]_k(\mathscr{U}) = [c]_k(\varphi_z(\mathscr{U}))$.

4) For a big prequantized convex toric domain $X_\Omega \times S^1 \subset T^*\mathbb{R}^d \times S^1$, we have

$$[c]_k(X_\Omega \times S^1) = \min\left\{p \in \mathbb{P} : \exists z \in \Omega_p^\circ, I(z) \geq k\right\} = \min\left\{p \in \mathbb{P} : p \geq c_k(X_\Omega)\right\}.$$

A more concrete example is as follows. Suppose $X_\Omega \times S^1 = E(3,4) \times S^1$, we have.

k	1	2	3	4	5	6	7	8	9	10	11
c_k	3	4	6	8	9	12	12	15	16	18	20
$[c]_k$	3	5	7	11	11	13	13	17	17	19	23

0.3. Microlocal theory of sheaves and the Chiu-Tamarkin complex

The main ingredient of our work is the microlocal theory of sheaves, introduced by Kashiwara and Schapira with motivation from algebraic analysis. We refer to [KS90].

The main idea we use in the paper is the notion of microsupport or singular support, which is defined as follows: For a ground commutative ring $\mathbb{K}$, let $D(X)$ be the derived category of complexes of sheaves of $\mathbb{K}$-modules over X. For an object $F \in D(X)$, we can associate a set $SS(F) \subset T^*X$, which is called the microsupport of F. It is proved in [KS90] that $SS(F)$ is always a closed conic and coisotropic subset of T^*X. Moreover, when X is real analytic, $SS(F)$ is Lagrangian if and only if F is (weakly) constructible. This result inspires us that the sheaf theory plays its role in symplectic geometry and contact geometry.

For instance, Tamarkin develops a new method to study displacibility of Lagrangians in [Tam18]. Guillermou gives sheaf theoretical proofs of Gromov-Eliashberg C^0-rigidity and of the result by Abouzaid and Kragh that closed exact Lagrangians in cotangent bundles are homotopically equivalent to the zero section. See [Gui12, Gui13, Gui16] and the survey [Gui23] about these topics. Asano-Ike develop a lower bound for the symplectic displacement energy and a lower bound for the number of intersection point of some rational Lagrangian immersions using numerical information from the Tamarkin Category (Definition 1.12) in [AI20a, AI20b]. On the other hand, there are many works studying the category of sheaves from the point of view of the Fukaya category, see the work of Nadler and Zaslow on the compact Fukaya category [NZ09, Nad09]; and the work of Nadler[Nad16], and of Ganatra, Pardon, and Shende on the wrapped Fukaya category[GPS18].

Now, let us review ideas of Tamarkin in [Tam18]. Tamarkin suggested studying the category of sheaves localized with respect to sheaves microsupported in non-positive direction, that is, the localization of $D(X \times \mathbb{R})$ with respect to the full thick subcategory $\{F : SS(F) \subset \{\tau \leq 0\}\}$. This localization is equivalent to the essential image of the functor $\mathbb{K}_{[0,\infty)}\star : D(X \times \mathbb{R}) \to D(X \times \mathbb{R})$, where $\star : D(X \times \mathbb{R}) \times D(X \times \mathbb{R}) \to D(X \times \mathbb{R})$ is the convolution. We denote these two equivalent categories by $\mathcal{D}(X)$ and call them the Tamarkin category of X. The category $\mathcal{D}(X)$ is triangulated.

Since the microsupport is conic, Tamarkin considers the following conification procedure: for a given closed set A in the cotangent bundle T^*X we set $\widehat{A} = \{(x, p, t, \tau) \in T^*(X \times \mathbb{R}) : (x, p/\tau) \in A, \tau > 0\}$. We are interested in the category of sheaves on $X \times \mathbb{R}$ microsupported in $\widehat{A}$, that is, $F \in \mathcal{D}(X)$ such that $SS(F) \cap \{\tau > 0\} \subset \widehat{A}$, and we denote the category they form by $\mathcal{D}_A(X)$. Categorically, this category and its semi-orthogonal complement ${}^\perp\mathcal{D}_A(X)$ are completely determined by the projectors from $\mathcal{D}(X)$ onto them. Hopefully, we could understand the geometry of A from these projectors. One way to study these projectors is to represent them as integral functors defined by kernels, for example $\mathbb{K}_{[0,\infty)}\star$ introduced by Tamarkin, or the *cut-off* functors of Kashiwara and Schapira [KS90].

Here, we start with the symplectic case. An open set $U \subset T^*X$ whose projector $\mathcal{D}(X) \to \mathcal{D}_U(X) =^\perp \mathcal{D}_{T^*X \setminus U}(X)$ is represented by a convolution functor $\star P_U : \mathcal{D}(X) \to \mathcal{D}(X)$ is called admissible. We will see later that bounded open sets and toric domains are all admissible. One particularly interesting example is the open ball $U = B_{\pi R^2}$. Chiu constructed a kernel for $B_{\pi R^2}$ using the idea of generating functions in [Chi17], which is the main ingredient of his proof of contact non-squeezing.

Another ingredient of Chiu's proof is (a contact version of) an object $C_T^{\mathbb{Z}/\ell}(U,\mathbb{K}) \in D_{\mathbb{Z}/\ell}(\mathrm{pt})$ defined using P_U, where $\mathbb{Z}/\ell$ is the cyclic group, $\mathbb{K}$ is a field, and $D_{\mathbb{Z}/\ell}(\mathrm{pt})$ denotes the equivariant derived category over point (see [BL94]). Remark that Chiu denotes our ℓ as N.

Chiu did not define the symplectic version $C_T^{\mathbb{Z}/\ell}(U,\mathbb{K})$ explicitly, while his idea applies directly to defining the symplectic version we presented in Definition 2.13. His discussions on contact invariance and computation work perfectly to the symplectic case as we will present in the following. We have that the object $C_T^{\mathbb{Z}/\ell}(U,\mathbb{K})$ is a Hamiltonian invariant of an admissible open set U for $\ell \in \mathbb{N}$ and $T \geq 0$. We will define a fundamental class $\eta_T^{\mathbb{Z}/\ell}(U,\mathbb{K}) \in H^0 C_T^{\mathbb{Z}/\ell}(U,\mathbb{K})$, and also see that $H^* C_T^{\mathbb{Z}/\ell}(U,\mathbb{K})$ is a left graded module over the Yoneda algebra $A = \mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{K},\mathbb{K})$. If $\mathrm{char}(\mathbb{K})|\ell$, we have $A \cong \mathbb{K}[u,\theta]$ where $|u| = 2$, $|\theta| = 1$, and $u^2 = k\theta$ (k depends on the parity of ℓ). To achieve the condition $\mathrm{char}(\mathbb{K})|\ell$, we can take $\mathbb{K} = \mathbb{F}_{p_\ell}$ to be the finite field of order p_ℓ where p_ℓ is the minimal prime factor of ℓ.

We will also discuss, in section 4, the contact version $\mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(\mathscr{U},\mathbb{K})$ that Chiu originally defines in [Chi17], for a contact admissible open set $\mathscr{U} \subset T^*X \times S^1$ and $(n,\ell) \in \mathbb{N}_0 \times \mathbb{N}$. The differences are that $\mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(\mathscr{U},\mathbb{K})$ is defined for $(n,\ell) \in \mathbb{N}_0 \times \mathbb{N}$ while $C_T^{\mathbb{Z}/\ell}(U,\mathbb{K})$ is defined for $(T,\ell) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ and that $\mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(\mathscr{U},\mathbb{K})$ is invariant under contact isotopies. The fundamental class $\eta_{n\ell}^{\mathbb{Z}/\ell,c}(\mathscr{U},\mathbb{K}) \in H^0\mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(\mathscr{U},\mathbb{K})$ and the A-module structure can also be defined. We will see that if the open set $U \subset T^*X$ is symplectic admissible, the prequantized open set $\mathscr{U} = U \times S^1 \subset T^*X \times S^1$ is contact admissible, and we have an isomorphism $\mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(U \times S^1,\mathbb{K}) \cong C_{n\ell}^{\mathbb{Z}/\ell}(U,\mathbb{K})$, which preserves the fundamental class. The isomorphism is helpful since even though the symplectic version is a priori not a contact invariant, it computes the contact version. In this sense, Chiu computed $C_\ell^{\mathbb{Z}/\ell}(B_{\pi R^2},\mathbb{K})$ using $\mathscr{C}_\ell^{\mathbb{Z}/\ell}(B_{\pi R^2} \times S^1,\mathbb{K})$.

Chiu's proof for the contact non-squeezing theorem can be organized by our language in the following steps:

- When ℓ is an odd prime number and $\pi r^2 > 1$, Chiu constructs an isomorphism of A-modules:

$$H^*\mathscr{C}_\ell^{\mathbb{Z}/\ell}(B_{\pi r^2} \times S^1,\mathbb{F}_\ell) \cong u^{-d\lfloor \ell/\pi r^2 \rfloor}\mathbb{F}_\ell[u,\theta],$$

and an element $\Lambda_r = ku^{-d\lfloor \ell/\pi r^2 \rfloor}$ such that $\eta_\ell^{\mathbb{Z}/\ell,c}(B_{\pi r^2} \times S^1,\mathbb{F}_\ell) = u^{d\lfloor \ell/\pi r^2 \rfloor}\Lambda_r \neq 0$.

• The fundamental class is preserved under the contact invariance. Specifically, for a compactly supported contact isotopy $\varphi : I \times T^*\mathbb{R}^d \times S^1 \to T^*\mathbb{R}^d \times S^1$ and $z \in I$, we have an isomorphism of A-modules

$$\Phi_{z\ell}^{\mathbb{Z}/\ell,c} : H^*\mathscr{C}_\ell^{\mathbb{Z}/\ell}(\varphi_z(B_{\pi r^2} \times S^1), \mathbb{F}_\ell) \cong H^*\mathscr{C}_\ell^{\mathbb{Z}/\ell}(B_{\pi r^2} \times S^1, \mathbb{F}_\ell)$$

such that $\eta_\ell^{\mathbb{Z}/\ell,c}(\varphi_z(B_{\pi r^2} \times S^1), \mathbb{F}_\ell)$ is mapped to $\eta_\ell^{\mathbb{Z}/\ell,c}(B_{\pi r^2} \times S^1, \mathbb{F}_\ell)$.

• If there exists an inclusion $B_{\pi R^2} \times S^1 \subset B_{\pi r^2} \times S^1$, for $R > r$, we have a degree 0 morphism of A-modules

$$i : H^*\mathscr{C}_\ell^{\mathbb{Z}/\ell}(B_{\pi r^2} \times S^1, \mathbb{F}_\ell) \to H^*\mathscr{C}_\ell^{\mathbb{Z}/\ell}(B_{\pi R^2} \times S^1, \mathbb{F}_\ell),$$

which preserves the fundamental class. In particular, we have $\eta_\ell^{\mathbb{Z}/\ell,c}(B_{\pi R^2} \times S^1, \mathbb{F}_\ell) = u^{d\lfloor \ell/\pi r^2 \rfloor} i(\Lambda_r)$ in $H^0\mathscr{C}_\ell^{\mathbb{Z}/\ell}(B_{\pi R^2} \times S^1, \mathbb{F}_\ell)$.

• However, the degree comparison makes $i(\Lambda_r) = 0$ for large enough ℓ. This is a contradiction because we know that $\eta_\ell^{\mathbb{Z}/\ell,c}(B_{\pi R^2} \times S^1, \mathbb{F}_\ell) \neq 0$.

We use the fundamental class, the module structure, and the invariance to define the capacities as we presented in the last two subsections. In this sense, our capacities form a numerical package of Chiu's arguments. Meanwhile, our third main result generalizes Chiu's computation of the Chiu-Tamarkin complex for balls to convex toric domains $X_\Omega \subset \mathbb{C}^d = T^*\mathbb{R}^d$.

As we already know that if $\mathscr{U} = U \times S^1$, the computation for both symplectic case and contact case is essentially the same. So, it is enough to compute the symplectic version. We will construct a good kernel for X_Ω based on Chiu's construction and then compute the symplectic Chiu-Tamarkin complex $C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$. We will show the following structural theorem.

Theorem 0.H (Theorem 3.6). *For a convex toric domain $X_\Omega \subset T^*\mathbb{R}^d$, and $\ell \in \mathbb{N}_{\geq 2}$. If $0 \leq T < p_\ell/\|\Omega_1^\circ\|_\infty$, we have*

• *For each $Z \in \Omega_T^\circ$ (see (0.2)), the inclusion of the segment $\overline{OZ} \subset \Omega_T^\circ$ induces a decomposition of the fundamental class $\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}) = u^{I(Z)}\Lambda_{Z,\ell}$ for a non-torsion element $\Lambda_{Z,\ell} \in H^{-2I(Z)}C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$. In particular, $\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ is non-zero.*

• *The minimal cohomology degree of $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ is exactly $-2I(\Omega_T^\circ)$, i.e.,*

$$H^*C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}) \cong H^{\geq -2I(\Omega_T^\circ)}C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}),$$

and

$$H^{-2I(\Omega_T^\circ)}C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}) \neq 0.$$

- $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ *is a finitely generated* $\mathbb{F}_{p_\ell}[u]$*-module. The free part is isomorphic to* $A = \mathbb{F}_{p_\ell}[u, \theta]$*, so* $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ *is of rank* 2 *over* $\mathbb{F}_{p_\ell}[u]$*.*

The torsion part is located in cohomology degree $[-2I(\Omega_T^\circ), -1]$*.* $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ *is torsion free when* X_Ω *is an open ellipsoid.*

0.4. Related works

In [FSZ23], we construct a $\mathbb{Z}/\ell$-equivariant generating function homology theory for $\ell \geq 2$ using a similar idea, and give a new proof of the contact non-squeezing theorem; we also define a geometric notion *translated chains*, which explains the geometry intuition behind of $\mathbb{Z}/\ell$-equivariant generating function homology. The notion of translated chains explains the same geometric intuition in the Chiu-Tamarkin complex.

Algebraically, in [Zha23], we construct an S^1-equivariant Chiu-Tamarkin complex which is similar to the Tsygan-Loday-Quillen definition of the cyclic cohomology. In particular, we construct an algebraic S^1-action that encodes $C_T^{\mathbb{Z}/\ell}(U, \mathbb{K})$ for all ℓ and $T \in \mathbb{R}_{\geq 0}$. However, if we want to define a contact invariant in this way, we need again $T/\ell \in \mathbb{N}_0$ for all ℓ, which is possible only for $n = 0$. It explains algebraically why we cannot define an S^1-theory for the contact case using the same idea. However, the $\mathbb{Z}/\ell$-theory here works perfectly for the contact case. The contact capacities we defined above encode sufficient numerical information that is enough for the contact non-squeezing theorem. In this way, we can think of the contact capacities as a numerical approximation of the S^1-action.

0.5. Organization and conventions of the paper

We will review preliminary notions of sheaf theory in section 1. In section 2, we will present the main constructions, including microlocal kernels, the Chiu-Tamarkin complex, fundamental class and capacities. We will focus on the toric domains in section 3. We would like to exhibit all constructions and computations for the toric domains therein. Subsequently, we will state how our construction works for prequantized contact manifold $T^*X \times S^1_\sigma$ in section 4.

At the end of the introduction, let us introduce some notation.

The natural number set $\mathbb{N}$ starts from 1, and $\mathbb{N}_0$ denotes $\mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, we denote $[n] = \{1, \dots, n\}$. For any $\ell \in \mathbb{N}_{\geq 2}$, we denote the minimal prime factor of ℓ by p_ℓ.

We use subscripts to represent elements in sets. For example, to emphasize $a \in A$, we use the notation A_a. For the Cartesian product A^n, we define $\delta_{A^n} : A \to A^n$ to be the diagonal map and its image is denoted by Δ_{A^n} as well.

Projection maps are always denoted by π, with a subscript that encodes the fiber of the projection. For example, if there are two sets X_x and Y_y, two projections are

$$\pi_Y = \pi_y : X_x \times Y_y \to X_x, \quad \pi_X = \pi_x : X_x \times Y_y \to Y_y.$$

If we have a trivial vector bundle $X \times V_v$, its summation map is

$$\begin{gathered}\mathrm{id}_X \times s_V^n = \mathrm{id}_X \times s_v^n : X \times V^n \to X \times V,\\ (x, v_1, \dots, v_n) \mapsto (x, v_1 + \cdots + v_n).\end{gathered}$$

In all cases, we will ignore id_X and only use $s_V^n = s_v^n$ for simplicity.

For a manifold X, we always use $\mathbf{q} \in X$ to represent both the points and the local coordinates of X. Correspondingly, the canonical Darboux coordinate of T^*X will be denoted by $(\mathbf{q}, \mathbf{p})$. Vector spaces that are considered as parameter spaces are an exception. For example, $\mathbb{R}_t$, its dual coordinate is denoted by $\tau \in (\mathbb{R}_t)^* = \mathbb{R}_\tau$.

For a manifold X, the 1-jet space is $J^1X = T^*X \times \mathbb{R}_t$, which is a contact manifold equipped with the contact form $\alpha = dt + \mathbf{p}d\mathbf{q}$. The symplectization of J^1X is identified with $T^*X \times T^*_{\tau>0}\mathbb{R}_t = T^*X \times \mathbb{R}_t \times \mathbb{R}_{\tau>0}$, equipped with the symplectic form $\omega = d\mathbf{p} \wedge d\mathbf{q} + d\tau \wedge dt$. The symplectic reduction of $T^*X \times T^*_{\tau>0}\mathbb{R}_t$ with respect to the hypersurface $\{\tau = 1\}$ is denoted by ρ, which is identified with

$$\rho : T^*X \times T^*_{\tau>0}\mathbb{R}_t \to T^*X,\ (\mathbf{q}, \mathbf{p}, t, \tau) \mapsto (\mathbf{q}, \mathbf{p}/\tau). \tag{0.4}$$

We call it the Tamarkin's cone map. The map ρ factors through the symplectization map q tautologically:

$$T^*X \times T^*_{\tau>0}\mathbb{R}_t \xrightarrow{\ q\ } J^1X \longrightarrow T^*X, \qquad \rho = (J^1X \to T^*X) \circ q.$$

In this paper, we equip the prequantized manifold $T^*X \times S^1_\sigma$ with the contact form $\alpha = d\sigma + \mathbf{p}d\mathbf{q}$, which is different with the contact form

$d\sigma + \frac{1}{2}(\mathbf{q}d\mathbf{p} - \mathbf{p}d\mathbf{q})$ we mentioned before. Then the canonical covering map $J^1X \to T^*X \times S^1_\sigma$ preserves its contact form.

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1. Reminder on sheaves and equivariant sheaves

In this section, we review the notions and tools of sheaves we will use. Let $\mathbb{K}$ be a commutative ring with finite global dimension. In practice, we only interest the case that $\mathbb{K}$ is a field or $\mathbb{K} = \mathbb{Z}$. For a manifold X, let us denote $D(X)$ the derived category of complexes of sheaves of $\mathbb{K}$-modules over X. We note that we do not specify the boundedness of complexes we used in general. In most of our applications, the complexes are locally bounded in the sense that their restrictions on relatively compact open sets are bounded. We refer to [KS90] as the main reference of the section.

1.1. Microsupport of sheaves and functorial estimate

For a locally closed inclusion $i : Z \subset X$ and $F \in D(X)$ we set

$$F_Z = i_! i^{-1} F, \quad \mathrm{R}\Gamma_Z F = i_* i^! F.$$

Definition 1.1 ([KS90, Definition 5.1.2]). For $F \in D(X)$ the microsupport of F is

$$SS(F) = \overline{\left\{ (\mathbf{q}, \mathbf{p}) \in T^*X : \begin{array}{l} \text{There is a } C^1\text{-function } f \text{ near } \mathbf{q} \text{ such that} \\ f(\mathbf{q}) = 0,\ df(\mathbf{q}) = \mathbf{p} \text{ and } \left(\mathrm{R}\Gamma_{\{f \geq 0\}} F\right)_{\mathbf{q}} \neq 0. \end{array} \right\}}.$$

By definition, $SS(F)$ is a closed subset of T^*X, conic with respect to the $\mathbb{R}_{>0}$-action along fibres. There is a triangulated inequality for the microsupport: for a distinguished triangle $A \to B \to C \xrightarrow{+1}$, we have $SS(A) \subset SS(B) \cup SS(C)$.

Theorem 1.2 ([KS90, Theorem 5.4.5(ii)(c)]). *For $F \in D(X)$, we have the equivalence:*

$$SS(F) \subset 0_X \textit{ if and only if } \forall k \in \mathbb{Z}, \mathcal{H}^k(F) \textit{ are local systems.}$$

We set $\dot{T}^*X = T^*X \setminus 0_X$, and $\dot{SS}(F) = SS(F) \cap \dot{T}^*X$.

The conicity is an issue since we want to consider general subsets of T^*X. We will use the Tamarkin's cone map ρ of (0.4) to resolve the conicity. This is important because most of symplectic geometric problems are nonconic. However, the cone map is only defined when $\tau > 0$ and it is helpful to introduce the Legendre microsupport and the sectional microsupport as follows: For sheaves $F \in D(X \times \mathbb{R}_t)$, we set

$$\begin{aligned} \mu s_L(F) &= q\left(SS(F) \cap \{\tau > 0\}\right) \subset J^1X, \\ \mu s(F) &= \rho\left(SS(F) \cap \{\tau > 0\}\right) \subset T^*X. \end{aligned} \tag{1.1}$$

A direct consequence is that $\mu s_L(F)$ and $\mu s(F)$ are not necessarily conic. However, $\mu s_L(F)$ and $\mu s(F)$ will lose $\tau \leq 0$ information. Usually, we will consider sheaves that satisfy $SS(F) \subset \{\tau \geq 0\}$ and it will often be the case, in practice, that $SS(F) \cap \{\tau \leq 0\} \subset 0_{X \times \mathbb{R}}$. So, Theorem 1.2 shows that we will not lose much information.

Let $f : X \to Y$ be a C^1 map of manifolds. Then there is a diagram of cotangent map:

$$T^*X \xleftarrow{df^*} X \times_Y T^*Y \xrightarrow{f_\pi} T^*Y$$

Definition 1.3. Let $f : X \to Y$ be a C^1 map of manifolds, and $\Lambda \subset T^*Y$ be a conic subset. One says that f is non-characteristic for Λ if for all $(\mathbf{q}, \mathbf{p}) \in \Lambda$ and $df^*_{\mathbf{q}}(\mathbf{p}) = 0$, we have $\mathbf{p} = 0$.

Then we list some functorial estimates we need.

Theorem 1.4 ([KS90, Theorem 5.4]). *Let $f : X \to Y$ be a C^1 map of manifolds, $F \in D(X), G \in D(Y)$. Let $\omega_{X/Y} = f^!\mathbb{K}_Y$ be the relative dualizing complex.*

1) One has

$$SS(F \overset{L}{\boxtimes} G) \subset SS(F) \times SS(G),$$
$$SS(\mathrm{R}\mathcal{H}om(\pi_X^{-1}F, \pi_Y^{-1}G)) \subset (-SS(F)) \times SS(G).$$

2) Assume f is proper on $\operatorname{supp}(F)$, *then $SS(\mathrm{R}f_!F) \subset f_\pi (df^*)^{-1} (SS(F))$.*

3) Assume f is non-characteristic for $SS(G)$. Then the natural morphism $f^{-1}G \overset{L}{\otimes} \omega_{X/Y} \to f^!G$ is an isomorphism, and $SS(f^{-1}G) \cup SS(f^!G) \subset df^* f_\pi^{-1} (SS(G))$.

4) Assume f is a submersion. Then $SS(F) \subset X \times_Y T^*Y$ if and only if $\forall j \in \mathbb{Z}$, the sheaves $\mathcal{H}^j(F)$ are locally constant on the fibres of f.

Corollary 1.5. *Let $F_1, F_2 \in D(X)$.*

1) Assume $SS(F_1) \cap (-SS(F_2)) \subset 0_X$, then $SS(F_1 \overset{L}{\otimes} F_2) \subset SS(F_1) + SS(F_2)$.

2) Assume $SS(F_1) \cap SS(F_2) \subset 0_X$, then $SS(\mathrm{R}\mathcal{H}om(F_2, F_1)) \subset (-SS(F_2)) + SS(F_1)$.

The following Corollary 1.6 is called the microlocal Morse lemma.

Corollary 1.6. *For $F \in D(X)$, let $\phi : X \to \mathbb{R}$ be a C^1-function that is proper on* $\operatorname{supp}(F)$. *Let $a < b$ in $\mathbb{R}$ and assume $d\phi(x) \notin SS(F)$ for $a \leq \phi(x) < b$. Then the natural morphisms $\mathrm{R}\Gamma(\{\phi(x) < b\}, F) \to \mathrm{R}\Gamma(\{\phi(x) < a\}, F)$ and $\mathrm{R}\Gamma_{\{\phi(x) \geq b\}}(X, F) \to \mathrm{R}\Gamma_{\{\phi(x) \geq a\}}(X, F)$ are isomorphisms.*

For the non-proper pushforward, we have

Theorem 1.7 ([Tam18, [Corollary 3.4]). *Let V be an $\mathbb{R}$-vector space, $\pi_V : X \times V \to X$, and $\pi_V^\# : T^*X \times V \times V^* \to T^*X \times V^*$ be the corresponding projections, and $i : T^*X \to T^*X \times V^*$ be the inclusion. Then for $F \in D(X \times V)$, we have*

$$SS(\pi_{V!}F), SS(\pi_{V*}F) \subset i^{-1}\overline{\pi_V^\#(SS(F))}.$$

1.2. Convolution and Tamarkin category

Let X_1, X_2, X_3 be three manifolds. Recall, $\pi_X : X \times Y \to Y$ is a projection whose fiber is X for arbitrary Y.

Definition 1.8. For $F \in D(X_1 \times X_2 \times \mathbb{R}_{t_1})$, $G \in D(X_2 \times X_3 \times \mathbb{R}_{t_2})$. The convolution is defined as

$$F \underset{X_2}{\star} G := \mathrm{R}s^2_{t!}\mathrm{R}\pi_{X_2!}(\pi^{-1}_{(X_3,t_2)}F \overset{L}{\otimes} \pi^{-1}_{(X_1,t_1)}G) \in D(X_1 \times X_3 \times \mathbb{R}_t).$$

In particular, when X_2 is a point, we use the notation $F_1 \boxtimes F_2$ to empathise. If there is no confusion, we can drop the subscript X_2 and write $F \star G = F \underset{X_2}{\star} G$ directly.

Similarly, for $F \in D(X_1 \times X_2)$, $G \in D(X_2 \times X_3)$, the composition is defined as

$$F \underset{X_2}{\circ} G = F \circ G := \mathrm{R}\pi_{X_2!}(\pi^{-1}_{X_3}F \overset{L}{\otimes} \pi^{-1}_{X_1}G) \in D(X_1 \times X_3).$$

For $0 \in \mathbb{R}$, $F \in D(X \times \mathbb{R})$, we have $\mathbb{K}_0 \star F \cong F$. So, the functor $\mathbb{K}_0\star$ plays the role of the identity functor. Besides, $\star$ and $\circ$ satisfy the following monoidal identities:

$$\begin{aligned} &(F_1 \star F_2) \star F_3 \cong F_1 \star (F_2 \star F_3), \quad (F_1 \circ F_2) \circ F_3 \cong F_1 \circ (F_2 \circ F_3), \\ &F_1 \star F_2 \cong F_2 \star F_1, \quad F_1 \circ F_2 \cong F_2 \circ F_1, \\ &(F_1 \star F_2) \circ F_3 \cong F_1 \star (F_2 \circ F_3). \end{aligned} \tag{1.2}$$

Here, the commutative identities are induced by the identification $X_1 \times X_3 \cong X_3 \times X_1$.

Remark 1.9. In specific cases, convolution could be presented by composition on $X \times \mathbb{R}_t$. For example, if $F \in D(X^2 \times \mathbb{R}_{t_1})$ and $G \in D(X \times \mathbb{R}_{t_2})$. Let $m(t, t') = t - t' = t_1$, then we have

$$F \star G \cong (m^{-1}F) \circ G.$$

In fact, taking $t' = t_2$, we can prove the isomorphism by the proper base change and the projection formula since $s^2_t(t_1, t_2) = t$.

However, convolution involves spaces of lower dimension. Therefore, we prefer to use convolution in this paper. More important, in geometric applications, the factor $\mathbb{R}_t$ will play the role of action. Then, convolutions are more helpful for us to look at action information.

Before going into further discussion, let us review the notion of semi-orthogonal decomposition of a triangulated category.

Let $\mathcal{T}$ be a triangulated category and $\mathcal{C}$ a thick full triangulated subcategory of $\mathcal{T}$. The left semi-orthogonal of $\mathcal{C}$ is defined by

$$^{\perp}\mathcal{C} := \{X \in \mathcal{T} : \mathrm{Hom}_{\mathcal{T}}(X, Y) = 0, \forall Y \in \mathcal{C}\}. \tag{1.3}$$

One can show that the following proposition holds, see [KS06, Chapter 4 and Exercise 10.15.].

Proposition 1.10. *Using the above notation, we have the following three equivalent properties:*

1) The inclusion $\mathcal{C} \to \mathcal{T}$ admits a left adjoint functor $L : \mathcal{T} \to \mathcal{C}$.

2) There is an equivalence $\mathcal{T}/\mathcal{C} \xrightarrow{\cong} {}^{\perp}\mathcal{C}$, where $\mathcal{T}/\mathcal{C}$ is the Verdier localization.

3) There are two functors $P, Q : \mathcal{T} \to \mathcal{T}$ such that $\forall X \in \mathcal{T}$, we have the distinguished triangle:

$$P(X) \to X \to Q(X) \xrightarrow{+1}$$

such that $P(X) \in {}^{\perp}\mathcal{C}$, and $Q(X) \in \mathcal{C}$.

In this situation, we say one of these data gives a left semi-orthogonal decomposition of $\mathcal{T}$. One can verify, if one of the conditions is satisfied, that $P^2 \cong P$, and $Q^2 \cong Q$. P, Q are called a pair of projectors associated to $\mathcal{C}$.

Now, let $\mathcal{T} = D(X \times \mathbb{R}_t)$, and $\mathcal{C} = \{F : SS(F) \subset \{\tau \leq 0\}\}$. The triangulated inequality of microsupport shows that $\mathcal{C}$ is a thick full triangulated subcategory of $\mathcal{T}$. Tamarkin constructs a pair of projectors associated to $\mathcal{C}$ given by convolution:

Theorem 1.11 ([Tam18]). *The functors $F \mapsto \mathbb{K}_{[0,\infty)} \star F$, $F \mapsto \mathbb{K}_{(0,\infty)}[1] \star F$ on $D(X \times \mathbb{R}_t)$ and the excision triangle,*

$$\mathbb{K}_{[0,\infty)} \to \mathbb{K}_0 \to \mathbb{K}_{(0,\infty)}[1] \xrightarrow{+1},$$

give a left semi-orthogonal decomposition of $D(X \times \mathbb{R}_t)$ associated to $\mathcal{C}$. Namely, for $F \in D(X \times \mathbb{R}_t)$ we have the distinguished triangle

$$\mathbb{K}_{[0,\infty)} \star F \to F \to \mathbb{K}_{(0,\infty)}[1] \star F \xrightarrow{+1}, \tag{1.4}$$

with $\mathbb{K}_{[0,\infty)} \star F \in {}^{\perp}\mathcal{C}$, $\mathbb{K}_{(0,\infty)}[1] \star F \in \mathcal{C}$.

One can also see [GS14, Proposition 4.19] for a proof and some generalizations of the proposition.

Definition 1.12. We define the Tamarkin category as the following left semi-orthogonal complement:

$$\mathcal{D}(X) = {}^{\perp}\{F : SS(F) \subset \{\tau \leq 0\}\} \cong D(X \times \mathbb{R})/\{F : SS(F) \subset \{\tau \leq 0\}\}.$$

By Proposition 1.10 and (1.4) , $F \in D(X \times \mathbb{R})$ is in $\mathcal{D}(X)$ if and only if

$$F \cong \mathbb{K}_{[0,\infty)} \underset{\mathrm{pt}}{\star} F \cong \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \underset{X}{\star} F. \tag{1.5}$$

Consequently, the convolution functor $\mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \underset{X}{\star}$ of the Tamarkin category $\mathcal{D}(X)$ coincides with the identity functor.

For $F \in \mathcal{D}(X)$, one can show $SS(F) \subset \{\tau \geq 0\}$ using microsupport estimates we mentioned last subsection, see [GS14, Proposition 4.17].

To build microlocal kernels, we follow Chiu's construction and use the Fourier-Sato transform, which is a sheaf-theoretic analogue of the Fourier transform. The Fourier-Sato transform defines a functor $D(V) \to D(V^*)$, where V is a real vector space and V^* is the dual of V. One can see [KS90, Section 3.7, Section 5.5] for more details. We mention that the Fourier-Sato transform gives an equivalence between $\mathbb{R}_{>0}$-equivariant sheaves on V and V^*. Tamarkin introduced a new version of the Fourier transform on the category $\mathcal{D}(V)$, which also works for non-$\mathbb{R}_{>0}$-equivariant sheaves. We call it the Fourier-Sato-Tamarkin transform. For the relation between the different versions of Fourier transforms, we refer to [D'A13, Gao17].

Let $Leg(V) = \{(z, \zeta, t) : t + \langle z, \zeta \rangle \geq 0\} \subset V \times V^* \times \mathbb{R}_t$, we consider $\mathbb{K}_{Leg(V)} \in \mathcal{D}(V \times V^*)$.

Definition 1.13. The Fourier-Sato-Tamarkin transform is defined as the functor

$$\begin{aligned} &\mathrm{FT} : D(X \times V_z \times \mathbb{R}) \to \mathcal{D}(X \times V^*_\zeta), \\ &\mathrm{FT}(F) = \widehat{F} := F \underset{V_z}{\star} \mathbb{K}_{Leg(V)}[\dim V]. \end{aligned}$$

One can see that the restriction of FT on $\mathcal{D}(X \times V_z)$ is an equivalence of categories in [Tam18, Theorem 3.5].

Sometimes, for $F \in D(V_\zeta^*)$, we will use the notation

$$\widehat{F} \coloneqq \mathbb{K}_{Leg(V)}[\dim V] \underset{V_\zeta^*}{\circ} F. \tag{1.6}$$

Geometrically, the set $Leg(V)$ is associated with the Legendre transform between V and V^*. The important thing for us is the microsupport estimate under the Fourier-Sato-Tamarkin transform. Combining Theorem 3.5 and Theorem 3.6 (and its proof) of [Tam18], we have

Theorem 1.14. *Let* $\varphi\colon J^1(X \times V) \to J^1(X \times V^*)$ *be the map* $\varphi(\boldsymbol{q}, \boldsymbol{p}, z, \zeta, t) = (\boldsymbol{q}, \boldsymbol{p}, \zeta, -z, t - \langle z, \zeta\rangle)$, *where we identify* V^{**} *with* V *naturally. Then for* $F \in \mathcal{D}(X \times V)$, *we have the microsupport relation:*

$$\mu s_L(\widehat{F}) = \varphi(\mu s_L(F)). \tag{1.7}$$

Proof. The original statement of [Tam18, Theorem 3.6] claim that $\mu s(\widehat{F}) \subset \varphi_0(\mu s(F))$, here $\varphi_0(\mathbf{q}, \mathbf{p}, z, \zeta) = (\mathbf{q}, \mathbf{p}, \zeta, -z)$. However, the proof indicates that the inclusion can be lifted to $J^1(X \times V)$ and φ, i.e.:

$$\mu s_L(\widehat{F}) \subset \varphi(\mu s_L(F)).$$

Moreover, Theorem 3.5 in loc. cit. shows that the Fourier transform $F \mapsto \widehat{F}$ has an inverse which is given by $G \mapsto \check{G} = G \underset{V_\zeta^*}{\star} \mathbb{K}_{Leg'(V)}$ where $Leg'(V) = \{(\zeta, z, t) : t \geq \langle z, \zeta\rangle\} \subset V^* \times V \times \mathbb{R}_t$. We also have an estimate

$$\mu s_L(\check{G}) \subset \varphi^{-1}(\mu s_L(G)).$$

Then the equal of (1.7) follows by taking $G = \widehat{F}$. □

1.3. Guillermou-Kashiwara-Schapira sheaf quantization

As a sheaf pattern of Hamiltonian action, we introduce the Guillermou-Kashiwara-Schapira (GKS for short) sheaf quantization as a basic tool here, see [GKS12] for more details.

Consider $\dot{T}^*Y$ as a symplectic manifold equipped with the Liouville symplectic form and with a $\mathbb{R}_{>0}$-action by dilation along the cotangent fibers. If $\varphi : I \times \dot{T}^*Y \to \dot{T}^*Y$ is a $\mathbb{R}_{>0}$-equivariant symplectic isotopy, one can show that it must be Hamiltonian with a $\mathbb{R}_{>0}$-equivariant Hamiltonian function H.

Consider its total graph

$$\Lambda_\varphi := \left\{(z, -H_z \circ \varphi_z(\mathbf{q},\mathbf{p}), (\mathbf{q},-\mathbf{p}), \varphi_z(\mathbf{q},\mathbf{p})) : (\mathbf{q},\mathbf{p}) \in \dot{T}^*Y, z \in I\right\}. \tag{1.8}$$

Then Guillermou, Kashiwara, and Schapira proved the following theorem:

Theorem 1.15 ([GKS12, Theorem 3.7]). *Using the above notation, we have a sheaf $K = K(\varphi) \in D(I \times Y^2)$ such that*
1) $\dot{SS}(K) = \Lambda_\varphi$,   2) $K_0 = \mathbb{K}_{\Delta_{Y^2}}$, where $K_z = K|_{\{z\}\times Y^2}$.

If we set $K_z^{-1} = v^{-1}\mathrm{R}\mathcal{H}om(K_z, \omega_Y \overset{L}{\boxtimes} \mathbb{K}_Y)$, $v(\boldsymbol{q}_1, \boldsymbol{q}_2) = (\boldsymbol{q}_2, \boldsymbol{q}_1)$, $\boldsymbol{q}_1, \boldsymbol{q}_2 \in Y$, $z \in I$, then

a) $\mathrm{supp}(K) \rightrightarrows I \times Y$ *are both proper,*

b) $K_z \circ K_z^{-1} \cong K_z^{-1} \circ K_z \cong \mathbb{K}_{\Delta_{Y^2}}$,

c) K is unique up to a unique isomorphism.

Consequently, $F \mapsto F \circ K_z$, $D(Y) \to D(Y)$ is an equivalence of categories for all $z \in I$, whose quasi inverse is $F \circ K_z^{-1}$.

For $F \in D(Y)$, $z_0 \in I$, we have

$$\dot{SS}(F \circ K_{z_0}) = \varphi_{z_0}(\dot{SS}(F)). \tag{1.9}$$

It means that, geometrically, $\circ K_z$ acts as the Hamiltonian isotopy φ.

Remark 1.16. Our convention for (1.8) is different from [GKS12]. Specifically, (1.8) is $\Lambda_{\varphi^{-1}}$ in loc. cit., and then our K_z should be K_z^{-1} in loc. cit. We choose such a convention for our convenience in adapting Chiu's proof for Proposition 2.8 without causing further consistency problems.

Let us describe two situations where we will use the theorem.

I) Let $\varphi : I \times T^*X \to T^*X$ be a compactly supported Hamiltonian isotopy. For $Y = X \times \mathbb{R}_t$, one can lift φ to $\widehat{\varphi} : I \times \dot{T}^*Y \to \dot{T}^*Y$. Specifically, we have the following:

Proposition 1.17 ([GKS12, Proposition A.6]). *Let $\varphi : I \times T^*X \to T^*X$ be a compactly supported Hamiltonian isotopy, whose Hamiltonian function is $H \in C^\infty(I \times T^*X)$.*

There is a $\mathbb{R}_{>0}$-equivariant Hamiltonian isotopy $\widehat{\varphi} : I \times \dot{T}^*Y \to \dot{T}^*Y$ such that:

a) *The function* $\widehat{H}(z, \boldsymbol{q}, t, \boldsymbol{p}, \tau) = \tau H(z, \boldsymbol{q}, \boldsymbol{p}/\tau)$ *is a Hamiltonian function of* $\widehat{\varphi}$ *on* $\{\tau \neq 0\}$.
b) *The lifting* $\widehat{\varphi}$ *commutes with both the symplectization and the Tamarkin's cone map.*
c) *We can take*

$$\begin{aligned} &\widehat{\varphi}(z, \boldsymbol{q}, t, \boldsymbol{p}, \tau) = (\tau \cdot \varphi(z, \boldsymbol{q}, \boldsymbol{p}/\tau), t - S_H(z, \boldsymbol{q}, \boldsymbol{p}/\tau)), && \tau \neq 0,\\ &\widehat{\varphi}(z, \boldsymbol{q}, t, \boldsymbol{p}, 0) = (\boldsymbol{q}, \boldsymbol{p}, t + v(z), 0), && \tau = 0, \end{aligned}$$

where $u \in C^\infty(I \times T^*X), v \in C^\infty(I)$, *and* $S_H(z, \boldsymbol{q}, \boldsymbol{p}) = \int_0^z [\alpha(X_{H_\lambda}) - H_\lambda] \circ \varphi_H^\lambda(\boldsymbol{q}, \boldsymbol{p}) d\lambda$ *is the symplectic action function.*
We call this $\widehat{\varphi}$ *or* $\widehat{\varphi}_z$ *the conification of* φ.

Remark 1.18. We notice that it is easy to lift φ to $T^*X \times T^*_{\tau>0}\mathbb{R}_t$ without the compactly supported assumption, however this is not enough to apply the Guillermou-Kashiwara-Schapira theorem. If we want to lift φ to $\dot{T}^*(X \times \mathbb{R}_t)$, we need the compactly supported condition.

Now, applying Theorem 1.15 to $\widehat{\varphi}$, we obtain a sheaf $K(\widehat{\varphi}) \in D(I \times X^2 \times \mathbb{R}_t^2)$.

In our later application, we prefer to use only one t-variable, and to use convolution. This is possible as follows. Consider $m(t_1, t_2) = t_2 - t_1$, then [Gui23, Corollary 2.3.2] shows that there is a unique $\mathcal{K}(\widehat{\varphi}) \in D(I \times X^2 \times \mathbb{R}_t)$ such that $K(\widehat{\varphi}) \cong m^{-1}\mathcal{K}(\widehat{\varphi})$, and $\mathcal{K}(\widehat{\varphi}) \cong \mathrm{R}m_! K(\widehat{\varphi})$. Then we can take $\mathcal{K}(\widehat{\varphi})$ as the sheaf quantization of φ. One can show that, for $F \in D(X \times \mathbb{R})$, we have $F \circ K(\widehat{\varphi}_z) \cong F \star \mathcal{K}(\widehat{\varphi}_z)$, see Remark 1.9.

By the commutativity of the lifting with symplectization, we have the following estimates for the Legendrian microsupport and sectional microsupport of $\mathcal{K}(\widehat{\varphi})$:

$$\begin{aligned} \mu s_L(\mathcal{K}(\widehat{\varphi})) \subset \{&(z, -H(\mathbf{q}, \mathbf{p}), \mathbf{q}, -\mathbf{p}, \varphi_z(\mathbf{q}, \mathbf{p}), -S_H(z, \mathbf{q}, \mathbf{p})) : \\ &(z, \mathbf{q}, \mathbf{p}) \in I \times T^*X\}, \\ \mu s(\mathcal{K}(\widehat{\varphi})) \subset \{&(z, -H(\mathbf{q}, \mathbf{p}), \mathbf{q}, -\mathbf{p}, \varphi_z(\mathbf{q}, \mathbf{p})) : \\ &(z, \mathbf{q}, \mathbf{p}) \in I \times T^*X\}. \end{aligned} \tag{1.10}$$

From the point of view of (1.9), for $F \in D(X \times \mathbb{R})$, we have

$$\mu s(F \star \mathcal{K}(\widehat{\varphi}_z)) = \mu s(F \circ K(\widehat{\varphi}_z)) = \varphi_z(\mu s(F)). \tag{1.11}$$

II) Let $\varphi : I \times T^*X \times S^1 \to T^*X \times S^1$ be a contact isotopy of $T^*X \times S^1$ with a contact Hamiltonian $H \in C^\infty(I \times T^*X \times S^1)$. One can lift φ to a

$\mathbb{Z}$-equivariant contact isotopy φ' of $J^1(X) = T^*X \times \mathbb{R}_t$, where $\mathbb{Z}$ acts by shifting t. Here, by $\mathbb{Z}$-equivariant, we mean that $J^1(\mathrm{T}_k)\varphi' = \varphi' J^1(\mathrm{T}_k)$ for $k \in \mathbb{Z}$, where $J^1(\mathrm{T}_k)(\mathbf{q}, \mathbf{p}, t) = (\mathbf{q}, \mathbf{p}, t + k)$.

Remark 1.19. In the symplectic case the Hamiltonian H does not depend on t, and does commute with T'_c for all real number c. In the contact case, φ' commute with T'_c only when for $c = k \in \mathbb{Z}$.

Then it is easy to lift φ' to the symplectization, $T^*X \times T^*_{\tau>0}\mathbb{R}_t$, of $J^1(X)$ to a $\mathbb{Z} \times \mathbb{R}_{>0}$ equivariant Hamiltonian isotopy $\widehat{\varphi'} : I \times T^*X \times T^*_{\tau>0}\mathbb{R}_t \to T^*X \times T^*_{\tau>0}\mathbb{R}_t$. Here, by $\mathbb{Z}$-equivariance, we mean that $d\mathrm{T}^*_k\widehat{\varphi'} = \widehat{\varphi'}d\mathrm{T}^*_k$ for $k \in \mathbb{Z}$, where $d\mathrm{T}^*_k(\mathbf{q}, \mathbf{p}, t, \tau) = (\mathbf{q}, \mathbf{p}, t + k, \tau)$ is the cotangent map of the shifting map $\mathrm{T}_k(\mathbf{q}, t) = (\mathbf{q}, t + k)$.

Similarly to the symplectic case, the compactly supported condition is necessary to extend $\widehat{\varphi'}$ to whole $\dot{T}^*(X \times \mathbb{R}_t)$.

In this case, we still take the sheaf quantization $K = K(\widehat{\varphi'}) \in D(I \times X^2 \times \mathbb{R}^2)$ of $\widehat{\varphi'}$ as sheaf quantization of φ. However now, since the contact Hamiltonian $H(\mathbf{q}, \mathbf{p}, t)$ will depend on the variable t, $K = K(\widehat{\varphi'})$ is not pulled back from $D(I \times X^2 \times \mathbb{R})$ by m. So, we will work with compositions rather than convolutions.

The $\mathbb{Z}$-equivariance is inherited by the sheaf $K(\widehat{\varphi'})$. Precisely, it means that

$$K(\widehat{\varphi'}) \circ \mathbb{K}_{\Delta_{X^2} \times \{(t,t+k):t\in\mathbb{R}\}} \cong \mathbb{K}_{\Delta_{X^2} \times \{(t,t+k):t\in\mathbb{R}\}} \circ K(\widehat{\varphi'}). \tag{1.12}$$

This is due to $\mathbb{K}_{\Delta_{X^2} \times \{(t,t+k):t\in\mathbb{R}\}} = \mathbb{K}_{\Gamma_{\mathrm{T}_k}}$ quantizes $d\mathrm{T}^*_k$, then we apply the uniqueness part of Theorem 1.15 to $\widehat{\varphi'} = d(\mathrm{T}_k^{-1})^*\widehat{\varphi'}d\mathrm{T}^*_k = d\mathrm{T}^*_{-k}\widehat{\varphi'}d\mathrm{T}^*_k$ to obtain the isomorphism (1.12).

1.4. Equivariant sheaves

Here, we review basic notions of equivariant sheaves. We refer to [BL94] for all details about the general theory of equivariant sheaves and equivariant derived categories. Suppose G is a compact Lie group. For a manifold X with a G action $\rho : G \times X \to X$, a G-equivariant sheaf is a pair (F, ψ) where $F \in Sh(X)$ and $\psi : \rho^{-1}F \cong \pi_G^{-1}F$ is an isomorphism of sheaves satisfying the cocycle conditions:

$$d_0^{-1}\psi \circ d_2^{-1}\psi = d_1^{-1}\psi, \quad s_0^{-1}\psi = \mathrm{Id}_F,$$

where

$$d_0(g,h,x) = (h, g^{-1}x), \quad d_1(g,h,x) = (gh, x),$$
$$d_2(g,h,x) = (g,x), \quad s_0(x) = (e,x).$$

A sheaf morphism between two G-equivariant sheaves is equivariant if it commutes with the ψ's. We set $Sh_G(X)$ be the category of G-equivariant sheaves. For example, when $X = \mathrm{pt}$, $Sh_G(X) \simeq \mathbb{K}[G] - \mathrm{Mod}$, the category of all G-modules. The category of G-equivariant sheaves $Sh_G(X)$ is abelian. Moreover, Grothendieck proved in [Gro57] that when G is finite, $Sh_G(X)$ admits enough injective objects. Therefore, the derived category $D(Sh_G(X))$ makes sense for finite groups, which is treated as a naive version of equivariant derived category of sheaves.

For general topological groups, the naive version is not good as our expectation. A basic difference is that $\mathrm{Ext}^*_{D(Sh_G(X))}(\mathbb{K}_X, \mathbb{K}_X)$ is not isomorphic to the equivariant cohomology of X. A more serious problem is how to define 6-operations with correct adjunction properties.

To resolve these problems, we must use the equivariant derived category $D_G(X)$ defined by Burnstein-Lunts, in where the expected isomorphism holds, and the correct 6-operations live.

For the compact Lie group G, there exists a universal bundle EG and a classifying space $BG = G\backslash EG$, which are unique up to homotopy. Now, we have a diagram of topological spaces:

$$X \xleftarrow{p} X \times EG \xrightarrow{q} X \times_G EG.$$

Definition 1.20. An object $F \in D_G(X)$ is a triple $F = (F_X, \overline{F}, \beta_F)$, where $F_X \in D(X)$, $\overline{F} \in D(X \times_G EG)$, and $\beta_F : p^{-1}F_X \to q^{-1}\overline{F}$ is an isomorphism in $D(X \times EG)$. A morphism $\alpha : F \to H$ is a pair $(\alpha_X, \overline{\alpha})$ where $\alpha_X : F_X \to H_X$, $\overline{\alpha} : \overline{F} \to \overline{H}$, and a commutative diagram in $D(X \times EG)$:

$$\begin{array}{ccc} p^{-1}F_X & \xrightarrow{\beta_F} & q^{-1}\overline{F} \\ {\scriptstyle p^{-1}\alpha_X}\downarrow & & \downarrow{\scriptstyle q^{-1}\overline{\alpha}} \\ p^{-1}H_X & \xrightarrow{\beta_H} & q^{-1}\overline{H}. \end{array}$$

For example, the equivariant constant sheaf is given by $\mathbb{K}_X^G = (\mathbb{K}_X, \mathbb{K}_{X \times_G EG}, \mathrm{Id}_{\mathbb{K}_{EG}})$. The natural functor $D_G(X) \to D(X \times_G EG)$, $F = (F_X, \overline{F}, \beta_F) \mapsto \overline{F}$ is fully faithful.

We have a forgetful functor $For : D_G(X) \to D(X)$ which is given by

$$F = (F_X, \overline{F}, \beta_F) \mapsto F_X.$$

In general, for Lie subgroups $H \subset G$, we have a restriction functor $For : D_G(X) \to D_H(X)$, and the forgetful functor correspondence to the case H trivial.

The microsupport of equivariant objects can be defined as follows:

Definition 1.21. For an object $F = (F_X, \overline{F}, \beta) \in D_G(X)$, where $F_X \in D(X)$, we *define* the microsupport of F to be $SS(F) := SS(F_X)$.

This definition makes sense since the contractibility of EG and Theorem 1.4-(4).

For a G-map $f : X \to Y$ between smooth manifolds, we define maps induced from f as follows:

$$\begin{array}{ccccc} X & \xleftarrow{p} & X \times EG & \xrightarrow{q} & X \times_G EG \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle \overline{f}} \\ Y & \xleftarrow[p']{} & Y \times EG & \xrightarrow[q']{} & Y \times_G EG. \end{array}$$

Then we can define 6-operations. For example, we have $\mathrm{R}f_*F = (\mathrm{R}f_*F_X, \mathrm{R}\overline{f}_*\bar{F}, \mathrm{R}\hat{f}_*\beta_F)$.

Proposition 1.22. *All properties of the 6-operations hold in the equivariant case. The 6-operations commute with the forgetful functor.*

Remark 1.23. Since the 6-operations commute with the forgetful functor, we will frequently use the notation of non-equivariant 6-operations to denote the equivariant 6-operations without explicit emphases.

In the equivariant derived category, we can obtain the expected isomorphism:

$$\begin{aligned} \mathrm{Ext}^*_{D_G(X)}(\mathbb{K}_X, \mathbb{K}_X) &\cong \mathrm{Ext}^*_{D(X\times_G EG)}(\mathbb{K}_{X\times_G EG}, \mathbb{K}_{X\times_G EG}) \\ &\cong H^*(X \times_G EG, \mathbb{K}). \end{aligned} \tag{1.13}$$

In particular, when $X = \mathrm{pt}$ is a point, we have

$$\mathrm{Ext}^*_{D_G(\mathrm{pt})}(\mathbb{K}, \mathbb{K}) \cong H^*(BG, \mathbb{K}). \tag{1.14}$$

For example,

$$\mathrm{Ext}^*_{D_{\mathbb{Z}/\ell}(\mathrm{pt})}(\mathbb{K},\mathbb{K}) \cong H^*(L_\ell^\infty,\mathbb{K}) \cong \mathbb{K}[u,\theta], \tag{1.15}$$

where $L_\ell^\infty = S^\infty/(\mathbb{Z}/\ell)$ is the infinite dimensional lens space, $\mathbb{K}$ is a finite field of $\mathrm{char}(\mathbb{K})|\ell$, $|u| = 2$, $|\theta| = 1$, and $\theta^2 = ku$ ($k = 0$ if ℓ is odd and $k = \ell/2$ otherwise). The computation can be found in [Hat02, Example 3E.2, Exercise 3E.1].

Remark 1.24. In [BL94], the authors use finite-dimensional approximations of EG to define 6-operations. The reason is, classically, the 6-operations and related propositions (especially the proper base change) are demonstrated for finite (cohomological) dimensional locally compact Hausdorff spaces while $X \times_G EG$ is not in this class. However, in the framework of [SS16], the authors introduce a relative notion called separated locally proper maps, for which a proper base change formula is true. In particular, our $\hat{f}$ and $\overline{f}$ are separated locally proper, and $\hat{f}_!$ and $\overline{f}_!$ have finite cohomological dimension. Consequently, we can provide simpler formula for the equivariant 6-operations, and they also work in the unbounded derived category.

For discrete groups G, both the naive and advanced versions are equivalent, i.e., $D(Sh_G(X)) \simeq D_G(X)$. In particular, $D(\mathbb{K}[G]-Mod) \simeq D_G(\mathrm{pt})$. In practice, we will use them alternatively without mentioning explicitly. As a rule of convenience, we only write a lower subscript G for all possible places to indicate that we are working on some version of equivariant categories.

2. Projectors, Chiu-Tamarkin complex, and capacities

2.1. Projectors associated to open sets in T^*X

In this subsection, we are going to study the categories related to sheaves microsupported in an open set $U \subset T^*X$. Next, we will construct kernels of the projectors onto these categories.

For a *closed* subset $Z \subset T^*X$ we define $\mathcal{D}_Z(X)$ as the full subcategory of $\mathcal{D}(X)$ consisting of the sheaves satisfying $\mu s(F) \subset Z$. For an *open* subset $U \subset T^*X$ we define $\mathcal{D}_U(X)$ to be the left semi-orthogonal complement of $\mathcal{D}_{T^*X\setminus U}(X)$ in $\mathcal{D}(X)$, i.e., $\mathcal{D}_U(X) = {}^\perp\mathcal{D}_{T^*X\setminus U}(X)$.

Now we have a diagram of inclusions

$$\mathcal{D}_{T^*X\setminus U}(X) \hookrightarrow \mathcal{D}(X) \hookleftarrow \mathcal{D}_U(X) \tag{2.1}$$

Following Tamarkin and Chiu, we are looking for convolution kernels that represent microlocal projector functors and give the corresponding semi-orthogonal decomposition.

Definition 2.1. We say U is $\mathbb{K}$-*admissible* if there is a distinguished triangle

$$P_U \to \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \to Q_U \xrightarrow{+1},$$

in $\mathcal{D}(X^2)$, such that the convolution functor $\star P_U$ is right adjoint to $\mathcal{D}_U(X) \hookrightarrow \mathcal{D}(X)$ and $\star Q_U$ is left adjoint to $\mathcal{D}_Z(X) \hookrightarrow \mathcal{D}(X)$, i.e.,

$$\mathcal{D}_Z(X) \xleftarrow{\star Q_U} \mathcal{D}(X) \xrightarrow{\star P_U} \mathcal{D}_U(X),$$

are two microlocal projectors.

Such a pair of sheaves (P_U, Q_U) together with the distinguished triangle give an orthogonal decomposition of $\mathcal{D}(X)$ by Proposition 1.10. We call the pair (P_U, Q_U) *microlocal kernels* associated with U, and the distinguished triangle as the defining triangle of U.

We say U is *admissible* if U is $\mathbb{Z}$-admissible.

Remark 2.2. 1) We define the $\mathbb{K}$-admissibility of U at the beginning. However the coefficient dependence seems redundant because our existence results in the following work for all $\mathbb{K}$, especially for $\mathbb{K} = \mathbb{Z}$. Moreover, one can show that if U is admissible, then U is $\mathbb{K}$-admissible for all $\mathbb{K}$ (by taking the tensor product $\mathbb{K}\overset{L}{\otimes}_{\mathbb{Z}}$ with kernels and then use the uniqueness below). From this point of view, we do not emphasize the coefficient ring $\mathbb{K}$ for the kernels (P_U, Q_U). We will see later that the $\mathbb{K}$ does affect the computation of the Chiu-Tamarkin complex.

2) The adjoint functors of the inclusion functor $\mathcal{D}_{T^*X\setminus U}(X) \hookrightarrow \mathcal{D}(X)$ is also studied in [Kuo23]. The author constructs the left and right adjoint of the inclusion functor, which are called infinite wrapping functors. Same with our existence results below (e.g. Corollary 2.10), the author also use the Guillermou-Kashiwara-Schapira sheaf quantization as a fundamental tool. Our point here is the existence of the kernel P_U.

In the following, we will present the functorial property, uniqueness of kernels, and existence of kernels for bounded open sets. Let us start with some basic facts.

Lemma 2.3. *Suppose $U_1 \subset U_2$ is an inclusion between $\mathbb{K}$-admissible open subsets in T^*X and their defining triangles are*

$$P_{U_i} \xrightarrow{a_i} \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \xrightarrow{b_i} Q_{U_i} \xrightarrow{+1}, \quad i = 1, 2.$$

1) We have $Q_{U_2} \star P_{U_1} \cong 0$, and the natural morphism

$$a_2 \star P_{U_1} = [P_{U_2} \star P_{U_1} \to P_{U_1}],$$

is an isomorphism. In particular, we have $P_U \star P_U \cong P_U$ and $Q_U \star P_U \cong 0$ for any admissible open set U.

2) For any admissible open set U and for all $F, G \in D(X^2 \times \mathbb{R})$, we have the isomorphism:

$$\mathrm{Hom}_{D(X^2 \times \mathbb{R})}(F \star P_U, G \star P_U) \to \mathrm{Hom}_{D(X^2 \times \mathbb{R})}(F \star P_U, G).$$

3) We have $\mathrm{RHom}(P_{U_1}, Q_{U_2}) \cong 0$ and

$$\mathrm{RHom}(P_{U_1}, a_2) : \mathrm{RHom}(P_{U_1}, P_{U_2}) \cong \mathrm{RHom}(P_{U_1}, \mathbb{K}_{\Delta_{X^2} \times [0,\infty)}). \tag{2.2}$$

Proof. (1) follows from the definition. For (2), consider the functor $\widetilde{\mathcal{P}}(F) = F \star P_U : D(X^2 \times \mathbb{R}) \to D(X^2 \times \mathbb{R})$, which is a projector on $D(X^2 \times \mathbb{R})^{op}$ in the sense of [KS06, Definition 4.1.1]. Notice that, the functor $\widetilde{\mathcal{P}}$ has the same formula as the microlocal projector but they have different domains. Then (2) follows from Proposition 4.1.3 in loc.cit.. For the vanishing of (3), we take $U = U_1$, $F = \mathbb{K}_{\Delta_{X^2} \times \{0\}}$, and $G = Q_{U_2}[d]$ for all $d \in \mathbb{Z}$ in (2). Next, applying $\mathrm{RHom}(P_{U_1}, -)$ to the defining triangle of U_2, we have that $\mathrm{RHom}(P_{U_1}, a_2)$ is an isomorphism. □

The functorial property of microlocal kernels is proven in [Chi17, Theorem 4.7(2)] for the contact case, and the uniqueness appears in [Zha20, Section 4.6] for the symplectic case. Here, we prove a strong form of the functorial property of kernels, which ensures that the defining triangle is also functorial and unique.

Proposition 2.4. *For any inclusion $U_1 \subset U_2 \subset T^*X$ between $\mathbb{K}$-admissible open subsets and their defining triangles*

$$P_{U_i} \xrightarrow{a_i} \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \xrightarrow{b_i} Q_{U_i} \xrightarrow{+1}, \quad i = 1, 2,$$

we have a morphism between the defining triangles:

$$\begin{array}{ccccccc} P_{U_1} & \xrightarrow{a_1} & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} & \xrightarrow{b_1} & Q_{U_1} & \xrightarrow{+1} & \\ \big\downarrow a & & \big\downarrow \mathrm{Id} & & \big\downarrow b & & \\ P_{U_2} & \xrightarrow{a_2} & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} & \xrightarrow{b_2} & Q_{U_2} & \xrightarrow{+1} & . \end{array}$$

These morphisms a, b are natural with respect to inclusions of admissible open sets. In particular, when $U_1 = U_2$ (while P_{U_1} and P_{U_2} are a priori not the same), the morphism of the defining triangles is an isomorphism of distinguished triangles.

Proof. The construction of a, b can be found in [Chi17, Theorem 4.7(2)]. He verifies that $a_1 = a_2 a$ and $b_2 = b b_1$ and a, b are natural with respect to inclusions. However a priori, a, b do not give a morphism of distinguished triangle.

So, we consider the following morphism of distinguished triangles constructed by TR3:

$$\begin{array}{ccccccc} P_{U_1} & \xrightarrow{a_1} & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} & \xrightarrow{b_1} & Q_{U_1} & \xrightarrow{+1} & \\ \big\downarrow a & & \big\downarrow \mathrm{Id} & & \big\downarrow \psi & & \\ P_{U_2} & \xrightarrow{a_2} & \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} & \xrightarrow{b_2} & Q_{U_2} & \xrightarrow{+1} & . \end{array}$$

Then we have $\psi b_1 = b_2$. As a corollary of Lemma 2.3-(3), we have the isomorphism:

$$\mathrm{Hom}(Q_{U_1}, Q_{U_2}) \xrightarrow{-\circ b_1} \mathrm{Hom}(\mathbb{K}_{\Delta_{X^2}\times[0,\infty)}, Q_{U_2}).$$

Finally, one conclude that $b = \psi$ since their image under the isomorphism $-\circ b_1$ is b_2. $\square$

By the results of [GKS12] recalled in subsection 1.3, there exists $\mathcal{K} \in D(X^2 \times \mathbb{R}_t)$ (taking $\mathcal{K} = \mathcal{K}(\widehat{\varphi})_z$) such that the convolution functor

$$D(X \times \mathbb{R}_t) \to D(X \times \mathbb{R}_t), \quad F \mapsto F \star \mathcal{K},$$

is an equivalence of categories and $\mu s_L(F \star \mathcal{K}) = \varphi_z(\mu s_L(F))$. Since $\widehat{\varphi}_z$ preserves τ of $T^*(X \times \mathbb{R}_t)$, this functor descends to the quotient $\mathcal{D}(X)$ and gives an auto-equivalence.

Corollary 2.5. *[Chi17, Proposition 4.5]Let $U \subset T^*X$ be an admissible open set, and φ be a compactly supported Hamiltonian isotopy $\varphi : I \times$*

$T^*X \to T^*X$. Then $\varphi_z(U)$ is admissible for all $z \in I$ and we have an isomorphism $P_{\varphi_z(U)} \cong \mathcal{K}^{-1} \star P_U \star \mathcal{K}$.

In particular, for $U = T^*X$, the isomorphism is realized by:

$$\mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \cong \mathcal{K}^{-1} \star \mathcal{K} \star \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \cong \mathcal{K}^{-1} \star \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \star \mathcal{K}.$$

On the other hand, we can also consider the rescaling of the size of U (or the rescaling of the symplectic form). To be simpler, assume that $X = \mathbb{R}^d$ is a $\mathbb{R}$-vector space. Consider the map $R : X \times \mathbb{R} \to X \times \mathbb{R}$, $(\mathbf{q}, t) \mapsto (\mathbf{q}/r, t/r^2)$ for $r > 0$. Then Theorem 1.4-(2) shows that

$$SS(R_!F) = \{(r\mathbf{q}, \mathbf{p}/r, r^2 t, \tau/r^2) : (\mathbf{q}, \mathbf{p}, t, \tau) \in SS(F)\}.$$

Therefore, if $\mu s(F) \subset T^*X \setminus U$, we have $\mu s(R_!F) \subset T^*X \setminus rU$ since $\rho(r\mathbf{q}, \mathbf{p}/r, r^2t, \tau/r^2) = (r\mathbf{q}, (\mathbf{p}/r)/(\tau/r^2)) = (r\mathbf{q}, r(\mathbf{p}/\tau))$. We can directly verify that $R_! : D(X^2 \times \mathbb{R}_t) \to D(X^2 \times \mathbb{R}_t)$ induces an equivalence $R_! : \mathcal{D}_U(X) \xrightarrow{\cong} \mathcal{D}_{rU}(X)$ whose quasi inverse is $R_!^{-1}$. Then we have an isomorphism of functors $\star P_{rU} \cong R_!^{-1}(- \star P_U)R_!$. Moreover, we have

Corollary 2.6. *Let $U \subset T^*\mathbb{R}^d$ be an admissible open set, and $r > 0$. Then rU is admissible and we have an isomorphism $P_{rU} \cong \bar{R}_! P_U$ where $\bar{R}(\boldsymbol{q}, \boldsymbol{q}', t) = (\boldsymbol{q}/r, \boldsymbol{q}'/r, t/r^2)$.*

Proof. Obviously, we have a distinguished triangle

$$\bar{R}_! P_U \to \mathbb{K}_{\Delta_{X^2}\times[0,\infty)} \to \bar{R}_! Q_U \xrightarrow{+1} .$$

Let us show that it is a defining triangle of rU. If so, the result follows from the uniqueness.

Take $\mathcal{R} = \mathbb{K}_{\Gamma_R}$ and $\mathcal{R}^{-1} = \mathbb{K}_{\Gamma_{R^{-1}}}$ where Γ_g denotes the graph of g. Then we have an isomorphism of functors $\circ\mathcal{R} \cong R_!$. Be careful that the convolution kernels $\mathcal{R}, \mathcal{R}^{-1}$ are objects in $D((X \times \mathbb{R})^2)$, and they cannot descent to $\mathcal{D}(X^2)$. So, we are necessary to consider compositions rather than convolutions. Based on Remark 1.9, we take $\mathscr{P}_U = m^{-1}P_U$ and $\mathscr{Q}_U = m^{-1}Q_U$ to guarantee that we have isomorphisms of functors: $\circ\mathscr{P}_U \cong \star P_U$ and $\circ\mathscr{Q}_U \cong \star Q_U$. In the following, we only discuss $\mathscr{P}_U$, but all the rest are true for $\mathscr{Q}_U$.

The previous discussion shows that $\circ\mathscr{P}_{rU} \cong R_!^{-1}(- \circ \mathscr{P}_U)R_! \cong - \circ (\mathcal{R}^{-1} \circ \mathscr{P}_U \circ \mathcal{R})$ as functors. Noticed that $\circ$ here only represent composition of sheaves. On the other hand, we have isomorphisms of composition kernels: $\mathcal{R}^{-1} \circ \mathscr{P}_U \circ \mathcal{R} \cong (R \times R)_! \mathscr{P}_U$. It follows from the general fact that $A \circ \mathbb{K}_{\Gamma_g} \cong (\mathrm{Id} \times g)_! A$ and $\mathbb{K}_{\Gamma_{g^{-1}}} \circ A \cong (g \times \mathrm{Id})_! A$ for any A and g.

Then we conclude by the base change isomorphism $(R \times R)_! m^{-1} \cong m^{-1} \bar{R}_!$. □

Now, let us study the existence of admissible open sets $U \subset T^*X$. In general, we can take a smooth Hamiltonian function H such that $U = \{H < 1\}$ ([Lee03, Theorem 2.29]). Our tools to construct kernels are sheaf quantizations and the Fourier-Sato-Tamarkin transform.

For our later application for toric domains, let us state our idea in a more general form. Suppose there is a Hamiltonian $\mathbb{R}^m_z$-action on T^*X, i.e., a symplectic action $\varphi : \mathbb{R}^m_z \times T^*X \to T^*X$ with a moment map $\mu : T^*X \to (\mathbb{R}^m_z)^* = \mathbb{R}^m_\zeta$. Let us consider U of the form $\mu^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^m_\zeta$. We assume that there exists a *sheaf quantization* $\mathcal{K} \in \mathcal{D}(\mathbb{R}^m_z \times X^2)$ associated with the Hamiltonian action in the sense:

$$
\begin{aligned}
&\mathcal{K}|_{z=0} \cong \mathbb{K}_{\Delta_{X^2} \times [0,\infty)}, \\
&\mu s(\mathcal{K}) \subset \{(z, -\mu(\mathbf{q},\mathbf{p}), \mathbf{q}, -\mathbf{p}, \varphi_z(\mathbf{q},\mathbf{p})) : (z,\mathbf{q},\mathbf{p}) \in \mathbb{R}^m_z \times T^*X\}.
\end{aligned} \tag{2.3}
$$

Remark 2.7. One can see that when $m = 1$, it is exactly the single Hamiltonian situation $\mu = H$. Now, if we additionally assume that $\mu = H$ is compactly supported up to constant, then we can take $\mathcal{K} = \mathcal{K}(\widehat{\varphi^H}) \star \mathbb{K}_{[0,\infty)}$ as the sheaf quantization. Here, $\mathcal{K}(\widehat{\varphi^H})$ is introduced in subsection 1.3.

In fact, as [GKS12, Remark 3.9] discussed, if we have an Hamiltonian action of $\mathbb{R}^m$ with $m > 1$, the sheaf quantization exists if the action is compactly supported.

The sheaf $\mathcal{K}$ lives over the z-variable space. Intuitively, if we want to restrict the microsupport of some sheaves into $\Omega \subset \mathbb{R}^m_\zeta$, we need a sheaf transform to interchange z and ζ variables, which are dual to each other. Then we cut-off the support of the resulting sheaf in some way. This operation is classical in mechanics and thermodynamics, i.e. the Legendre transform. We have noticed that the sheaf correspondence of the Legendre transform is the Fourier-Sato(-Tamarkin) transform. Consequently, let us apply the Fourier-Sato-Tamarkin transform to the z-variable, i.e. $\widehat{\mathcal{K}} = \mathcal{K} \star \mathbb{K}_{Leg(\mathbb{R}^m_\zeta)}[m] \in \mathcal{D}(\mathbb{R}^m_\zeta \times X^2)$. So by (2.3) and (1.7), we have

$$
\mu s(\widehat{\mathcal{K}}) \subset \{(\mu(\mathbf{q},\mathbf{p}), z, \mathbf{q}, -\mathbf{p}, \varphi_z(\mathbf{q},\mathbf{p})) : (z,\mathbf{q},\mathbf{p}) \in \mathbb{R}^m_z \times T^*X\}. \tag{2.4}
$$

Then, we construct the kernels in the following way. Consider the excision triangle in $D(\mathbb{R}^m_\zeta)$:

$$\mathbb{K}_\Omega \to \mathbb{K}_{\mathbb{R}^m_\zeta} \to \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega} \xrightarrow{+1} .$$

Composing the distinguished triangle with $\widehat{\mathcal{K}}$, we obtain a distinguished triangle in $\mathcal{D}(X^2)$:

$$\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega} \xrightarrow{+1} .$$

By the associativity of convolutions and compositions (see (1.2)), we have $\widehat{\mathcal{K}} \circ F = (\mathcal{K} \star \mathbb{K}_{Leg(\mathbb{R}^m_\zeta)}[m]) \circ F \cong \mathcal{K} \star (\mathbb{K}_{Leg(\mathbb{R}^m_\zeta)}[m] \circ F) \cong \mathcal{K} \star \widehat{F}$ for $F \in D(\mathbb{R}^m_\zeta)$, where $\widehat{F} = \mathbb{K}_{Leg(\mathbb{R}^m_\zeta)}[m] \circ F$ in (1.6).

So as $\mathbb{K}_{Leg(\mathbb{R}^m_\zeta)}[m] \circ \mathbb{K}_{\mathbb{R}^m_\zeta} = \mathbb{K}_{\{z=0,\, t \geq 0\}}$, we have

$$(\widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta}) \cong \mathcal{K} \star \mathbb{K}_{\{z=0,\, t\geq 0\}} \cong \mathbb{K}_{\Delta_{X^2} \times [0,\infty)},$$

where the last isomorphism comes from (2.3), i.e., $\mathcal{K}|_{z=0} \cong \mathbb{K}_{\Delta_{X^2} \times [0,\infty)}$. Therefore, we have the distinguished triangle

$$\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \to \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega} \xrightarrow{+1} .$$

Proposition 2.8. *Let φ be a Hamiltonian $\mathbb{R}^m_z$-action on T^*X with a moment map $\mu : T^*X \to \mathbb{R}^m_\zeta$. We assume that there exists a sheaf quantization $\mathcal{K} \in D(\mathbb{R}^m_z \times X^2 \times \mathbb{R}_t)$ of the Hamiltonian action in the sense of* (2.3)*. For an open subset $\Omega \subset \mathbb{R}^m_\zeta$ such that for all $\zeta \in \Omega$, $\mu^{-1}(\zeta)$ is compact, the open set $U = \mu^{-1}(\Omega) \subset T^*X$ is admissible.*

More precisely, the pair of sheaves

$$P_U \coloneqq \widehat{\mathcal{K}} \circ \mathbb{K}_\Omega,\; Q_U \coloneqq \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega}, \tag{2.5}$$

and the distinguished triangle

$$\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \to \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^m_\zeta \setminus \Omega} \xrightarrow{+1}, \tag{2.6}$$

provide the microlocal kernels of U and the semi-orthogonal decomposition.

Proof. Our construction is a straightforward generalization of Chiu's result [Chi17, Theorem 3.11]. One only needs to notice that we consider a Hamiltonian $\mathbb{R}^m_z$-action more than a single Hamiltonian function, and we replace

$(-\infty, R)$ in Chiu's paper by Ω. The properness condition of Ω is a technical condition that is automatically true in the situation of Chiu. One can check the proof of Chiu to confirm that our condition is enough to ensure the virtue of the semi-orthogonal decomposition without any other modification. Furthermore, Chiu's argument works for $\mathbb{Z}$. So we obtain not only $\mathbb{K}$-admissibility but also admissibility. $\square$

Definition 2.9. We say an admissible open set U is *dynamically admissible* if there exists φ, $\mathcal{K}$, Ω that satisfies Proposition 2.8.

So, for dynamically admissible sets, Proposition 2.8 provides us with a standard way to construct microlocal kernels. Now, let us state some corollaries.

Corollary 2.10. *Bounded open sets are dynamically admissible.*

Proof. Let $U \subset T^*X$ be a bounded open set, we have $T^*X \setminus U$ is a closed subset of T^*X. Then there exists a smooth function $H : T^*X \to [0, 1]$ such that $U = \{H < 1\}$ and $T^*X \setminus U = \{H \geq 1\}$. Actually, we take a non negative function f such that $f^{-1}(0) = T^*X \setminus U$, see [Lee03, Theorem 2.29]. Then we take $H(x) = 1 - f(x)$.

Since U is bounded, the subsets $\{H = a\} \subset U$ with $a < 1$ are compact. Moreover, dH has compact support. So we can take the GKS quantization $\mathcal{K}(\widehat{\varphi^H})$. Then the result follows from Proposition 2.8 by taking $\Omega = (-\infty, 1)$. $\square$

The second corollary here concerns the kernel of products of open sets.

Corollary 2.11. *Suppose we have two bounded open sets $U_i \subset T^*X_i$, with two pairs of kernels (P_{U_i}, Q_{U_i}), $i = 1, 2$. Then $U_1 \times U_2$ is dynamically admissible and $P_{U_1 \times U_2} \cong P_{U_1} \boxtimes P_{U_2}$.*

Proof. By the assumption, we have two Hamiltonian functions $H_i \in C^\infty(T^*X_i)$ such that $U_i = \{H_i < 1\}$ and we associate with them two sheaf quantizations $\mathcal{K}_i$. Then

$$(P_{U_i}, Q_{U_i}) = (\widehat{\mathcal{K}}_i \circ \mathbb{K}_{(-\infty,1)}, \widehat{\mathcal{K}}_i \circ \mathbb{K}_{[1,\infty)}), \quad i = 1, 2.$$

Now, consider the product Hamiltonian $\mathbb{R}^2_z$-action on $T^*(X_1 \times X_2)$ whose moment map is $\mu = (H_1, H_2)$. Then $\mathcal{K}_1 \boxtimes \mathcal{K}_2$ is a sheaf quantization of the Hamiltonian action in the sense of (2.3).

Observe that if we take $\Omega = \{\zeta = (\zeta_1, \zeta_2) : \zeta_1 < 1, \zeta_2 < 1\}$, then we have $U_1 \times U_2 = \mu^{-1}(\Omega)$. Consequently, Proposition 2.8 implies that $U_1 \times U_2$ is admissible by the following distinguished triangle

$$\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \to \mathbb{K}_{\Delta \times \{t \geq 0\}} \to \widehat{\mathcal{K}} \circ \mathbb{K}_{\mathbb{R}^2_\zeta \setminus \Omega} \xrightarrow{+1} .$$

Subsequently, let us compute $\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega$.

Recall

$$\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \cong \mathcal{K} \star \widehat{\mathbb{K}_\Omega}.$$

Notice Ω is an open convex set. Therefore, $\widehat{\mathbb{K}_\Omega} = \mathbb{K}_{\{(z,\zeta,t): t + z\cdot\zeta \geq 0\}}[2] \circ \mathbb{K}_\Omega$ is the constant sheaf $\mathbb{K}_{\Omega^\circ}$ supported on the polar cone Ω° of Ω, where

$$\Omega^\circ = \{(z,t) : t + z \cdot \zeta \geq 0, \forall \zeta \in \Omega\}.$$

In fact, $\mathbb{K}_{\{(z,\zeta,t): t + z\cdot\zeta \geq 0\}}[2] \circ \mathbb{K}_\Omega$ is isomorphic to the classical Fourier-Sato transform of the constant sheaf supported on the conification of Ω (upto a degree shifting depends only on dimension). The conification is a convex cone. The Fourier-Sato transform of a constant sheaf supported on a convex cone is the constant sheaf supported on the polar cone of the given cone. A direct computation shows that the polar cone of the conification of Ω is exactly Ω°. Then, our computation follows.

In particular, when $\Omega = \{\zeta_1 < 1, \zeta_2 < 1\}$, we have $\Omega^\circ = \{(z,t) : z = (z_1, z_2), , z_1 \leq 0, z_2 \leq 0, t \geq -(z_1 + z_2) \geq 0\}$. Moreover, $\mathbb{K}_{\Omega^\circ} \cong \mathrm{R}s^2_{t!}(\mathbb{K}_{\gamma_1 \times \gamma_2})$, where $\gamma_i = \{(z_i, t) : t \geq -z_i \geq 0\}$.

Now we have

$$\begin{aligned}\widehat{\mathcal{K}} \circ \mathbb{K}_\Omega &\cong \mathcal{K} \star \widehat{\mathbb{K}_\Omega} \cong \mathcal{K} \star \mathbb{K}_{\Omega^\circ} \cong \mathcal{K} \star \mathrm{R}s^2_{t!}(\mathbb{K}_{\gamma_1 \times \gamma_2}) \\ &\cong (\mathcal{K}_1 \boxtimes\!\!\!\!\star\; \mathcal{K}_2) \star \mathrm{R}s^2_{t!}(\mathbb{K}_{\gamma_1 \times \gamma_2}) \cong (\mathcal{K}_1 \star \mathbb{K}_{\gamma_1}) \boxtimes\!\!\!\!\star\; (\mathcal{K}_2 \star \mathbb{K}_{\gamma_2}).\end{aligned}$$

Finally, noticing that $\mathbb{K}_{\{(z,t): t \geq -z \geq 0\}} \cong \mathbb{K}_{\{(z,\zeta,t): t + z\zeta \geq 0\}}[1] \circ \mathbb{K}_{(-\infty,1)}$, one can conclude that

$$P_{U_1 \times U_2} \cong \widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \cong (\mathcal{K}_1 \star \mathbb{K}_{\gamma_1}) \boxtimes\!\!\!\!\star\; (\mathcal{K}_2 \star \mathbb{K}_{\gamma_2}) \cong P_{U_1} \boxtimes\!\!\!\!\star\; P_{U_2}.$$

□

2.2. Chiu-Tamarkin complex

Let $\mathbb{Z}/\ell$ be the finite cyclic group of order $\ell \in \mathbb{N}$ and X be a smooth manifold of dimension d.

Now take an admissible open set $U \subset T^*X$, and let P_U be the kernel associated with U. The manifold $(X^2 \times \mathbb{R}_t)^\ell$ admits a $\mathbb{Z}/\ell$-action induced by the cyclic permutation of the ℓ factors. According to [Lon21, Section 2.2], the object $P_U^{\overset{L}{\boxtimes}\ell}$ of $D((X^2 \times \mathbb{R}_t)^\ell)$ has a natural lift $St_D(P_U)$ as an object of the equivariant derived category $D_{\mathbb{Z}/\ell}((X^2 \times \mathbb{R}_t)^\ell)$, which we also denote, due to historical reason, by $P_U^{\overset{L}{\boxtimes}\ell}$. Then we have $P_U^{\boxast \ell} = \mathrm{R}s^\ell_{t!} P_U^{\overset{L}{\boxtimes}\ell} \in D_{\mathbb{Z}/\ell}((X^2)^\ell \times \mathbb{R}_t)$.

Consider the $\mathbb{Z}/\ell$-equivariant maps

$$
\begin{aligned}
\pi_{\underline{\mathbf{q}}} &: X^\ell \times \mathbb{R} \to \mathbb{R},\\
\tilde{\Delta}_X &: X^\ell \times \mathbb{R} \to X^{2\ell} \times \mathbb{R},\\
\tilde{\Delta}_X(\mathbf{q}_1, \dots, \mathbf{q}_\ell, t) &= (\mathbf{q}_\ell, \mathbf{q}_1, \mathbf{q}_1, \dots, \mathbf{q}_{\ell-1}, \mathbf{q}_{\ell-1}, \mathbf{q}_\ell, t),\\
i_T &: \{T\} \hookrightarrow \mathbb{R},
\end{aligned}
$$

where $\underline{\mathbf{q}} = (\mathbf{q}_1, \dots, \mathbf{q}_\ell)$ and $\tilde{\Delta}_X$ is a twisted diagonal map of X.

There is an adjoint pair $(\alpha_{\ell,T,X}, \beta_{\ell,T,X})$:

$$
F \in D_{\mathbb{Z}/\ell}((X^2 \times \mathbb{R}_t)^\ell) \underset{\beta_{\ell,T,X}}{\overset{\alpha_{\ell,T,X}}{\rightleftarrows}} D_{\mathbb{Z}/\ell}(\mathrm{pt}) \ni G,
$$

defined by:

$$
\begin{aligned}
\alpha_{\ell,T,X}(F) &= i_T^{-1}\mathrm{R}\pi_{\underline{\mathbf{q}}!}\tilde{\Delta}_X^{-1}\mathrm{R}s^\ell_{t!}(F),\\
\beta_{\ell,T,X}(G) &= s_t^{\ell!}\tilde{\Delta}_{X*}\pi^!_{\underline{\mathbf{q}}} i_{T*}G.
\end{aligned}
\tag{2.7}
$$

Now, we define a functor

$$
F_{\ell,X} = \mathrm{R}\pi_{\underline{\mathbf{q}}!}\tilde{\Delta}_X^{-1}\mathrm{R}s^\ell_{t!} : D_{\mathbb{Z}/\ell}((X^2 \times \mathbb{R}_t)^\ell) \to D_{\mathbb{Z}/\ell}(\mathbb{R}).
\tag{2.8}
$$

Then $\alpha_{\ell,T,X} = i_T^{-1}F_{\ell,X}$.

Similarly, we define $\alpha'_{\ell,T,X}, \beta'_{\ell,T,X}, F'_{\ell,T,X}$ by removing s_t^ℓ in the corresponding definitions. If there is no risk of confusion, forget some of ℓ, T, X in subscripts of α, β, F for simplicity.

Remark 2.12. We will use $\alpha_{\ell,T,X}, \beta_{\ell,T,X}$ $(\alpha'_{\ell,T,X}, \beta'_{\ell,T,X})$, and $F_{\ell,X}$ $(F'_{\ell,X})$ in the non-equivariant categories. We denote them by the same notation later.

Definition 2.13. With the notation above, we first define

$$F_\ell(U,\mathbb{K}) := F_{\ell,X}(P_U^{\overset{L}{\boxtimes}\ell}) = F'_{\ell,X}(P_U^{\boxast\ell}) \in D_{\mathbb{Z}/\ell}(\mathbb{R}).$$

Then we define an object of $D_{\mathbb{Z}/\ell}(\mathrm{pt})$ that we call the *Chiu-Tamarkin complex* by

$$\begin{aligned} C_T^{\mathbb{Z}/\ell}(U,\mathbb{K}) &= \mathrm{RHom}_{\mathbb{Z}/\ell}\left(\alpha_{\ell,T,X}(P_U^{\overset{L}{\boxtimes}\ell}), \mathbb{K}[-d]\right) \\ &= \mathrm{RHom}_{\mathbb{Z}/\ell}\left((F_\ell(U,\mathbb{K}))_T, \mathbb{K}[-d]\right) \\ &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}\left(P_U^{\overset{L}{\boxtimes}\ell}, \beta_{\ell,T,X}\mathbb{K}[-d]\right). \end{aligned}$$

We set $A = \mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{K},\mathbb{K})$, which is isomorphic to $H^*_{\mathbb{Z}/\ell}(B\mathbb{Z}/\ell.\mathbb{K})$ (see (1.14)). Then $H^*C_T^{\mathbb{Z}/\ell}(U,\mathbb{K})$ is a graded module over $A \cong \mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{K}[-d],\mathbb{K}[-d])$ via the Yoneda product.

When $\ell = 1$, i.e. the cyclic group $\mathbb{Z}/1$ is trivial, we also denote the non-equivariant Chiu-Tamarkin complex $C_T^{\mathbb{Z}/1}(U,\mathbb{K})$ by $C_T(U,\mathbb{K})$.

Remark 2.14. 1) The object $C_T^{\mathbb{Z}/\ell}(U,\mathbb{K})$ is mentioned by Tamarkin in [Tam15], and is defined explicitly by Chiu in [Chi17]. Our definition looks slightly different from the definition of Chiu. However, one can check directly that, when X is orientable, $\beta_{\ell,T,X}\mathbb{K}[-d]$ is exactly the constant sheaf supported on the twisted diagonal with a degree shift depending only on ℓ and $\dim X$. So the complex $C_T^{\mathbb{Z}/\ell}(U,\mathbb{K})$ is essentially the same as what Chiu defined.

2) Recall Lemma 2.3, we have that, for all $\ell \in \mathbb{N}_{\geq 2}$, $P_U^{\star\ell} \cong P_U$ in $\mathcal{D}(X^2)$. Then we have $F_1(P_U^{\star\ell}) \cong F_1(P_U)$. However the definition of convolution shows $F_1(P_U^{\star\ell}) \cong F_\ell(P_U^{\overset{L}{\boxtimes}\ell})$. Therefore, we obtain an isomorphism, in the non-equivariant derived category,

$$F_1(U,\mathbb{K}) \cong F_\ell(U,\mathbb{K}). \tag{2.9}$$

So we have $C_T(U,\mathbb{K}) \cong \mathrm{RHom}\left((F_\ell(U,\mathbb{K}))_T, \mathbb{K}[-d]\right)$ (in the non-equivariant derived category). In this way, it is clear that $C_T^{\mathbb{Z}/\ell}(U,\mathbb{K})$ is the equivariant generalization of $C_T(U,\mathbb{K})$.

Let us compute an example when $U = T^*X$. Recall $P_{T^*X} = \mathbb{K}_{\Delta_{X^2}\times[0,\infty)}$, so we have $\tilde{\Delta}^{-1}\left(P_{T^*X}^{\boxtimes \ell}\right) = \mathbb{K}_{\Delta_{X^\ell}\times[0,\infty)}$. Then we obtain

$$F_\ell(T^*X, \mathbb{K}) = \mathrm{R}\pi_{\underline{\mathbf{q}}!}(\mathbb{K}_{\Delta_{X^\ell}\times[0,\infty)}) = E_{[0,\infty)},$$

where $E = R\Gamma_c(\Delta_{X^\ell}, \mathbb{K})$, $E_{[0,\infty)}$ is the constant sheaf supported on $[0,\infty)$ and $\mathbb{Z}/\ell$ acts on $E = \mathrm{R}\Gamma_c(\Delta_{X^\ell}, \mathbb{K}) \cong \mathrm{R}\Gamma_c(X, \mathbb{K})$ trivially. Since $\mathbb{Z}/\ell$ acts on E trivially, we have, by Poincaré-Verdier duality,

$$\begin{aligned} C_T^{\mathbb{Z}/\ell}(T^*X, \mathbb{K}) &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}(E, \mathbb{K}[-d]) \\ &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}(\mathbb{K}, \mathbb{K}) \overset{L}{\otimes} \mathrm{RHom}(E, \mathbb{K}[-d]) \\ &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}(\mathbb{K}, \mathbb{K}) \overset{L}{\otimes} \mathrm{R}\Gamma(X, \omega_X[d]) \\ &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}(\mathbb{K}, \mathbb{K}) \overset{L}{\otimes} \mathrm{R}\Gamma_X(T^*X, \mathbb{K})[d]. \end{aligned} \tag{2.10}$$

Finally, for $T \geq 0$ and a field $\mathbb{K}$, we have

$$H^*C_T^{\mathbb{Z}/\ell}(T^*X, \mathbb{K}) \cong A \otimes H_{d-*}^{BM}(X, \mathbb{K}) \cong A \otimes H_X^{*+d}(T^*X, \mathbb{K}), \tag{2.11}$$

where $H_*^{BM} = H^{-*}(X, \omega_X)$ stands for the Borel-Moore homology of X.

One of the most important theorems about the Chiu-Tamarkin complex is

Theorem 2.15 (Theorem 4.7 of [Chi17]). *Let U, U_1, U_2 be admissible open sets and let $U_1 \overset{i}{\hookrightarrow} U_2$ be an inclusion. Then one has, for $T \geq 0$,*

1) There is a morphism $C_T^{\mathbb{Z}/\ell}(U_2, \mathbb{K}) \xrightarrow{i^*} C_T^{\mathbb{Z}/\ell}(U_1, \mathbb{K})$, which is functorial with respect to inclusions of admissible open sets.

2) For a compactly supported Hamiltonian isotopy $\varphi : I \times T^*X \to T^*X$, then there is an isomorphism, in the equivariant category, $\Phi_{z,T}^{\mathbb{Z}/\ell}$: $C_T^{\mathbb{Z}/\ell}(U, \mathbb{K}) \xrightarrow{\cong} C_T^{\mathbb{Z}/\ell}(\varphi_z(U), \mathbb{K})$, for all $z \in I$. The isomorphism $\Phi_{z,T}^{\mathbb{Z}/\ell}$ is functorial with respect to the restriction morphisms in (1). When $U = T^*X$, we have $\Phi_{z,T}^{\mathbb{Z}/\ell} = \mathrm{Id}$.

Taking into account the structure of $A = \mathrm{Ext}_{\mathbb{Z}/\ell}^*(\mathbb{K}, \mathbb{K})$-modules, we have

Corollary 2.16. *With the notation of Theorem 2.15, we have:*

1) $H^*(i^*)$ is a morphism of A-modules.

2) $H^*(\Phi_{z,T}^{\mathbb{Z}/\ell})$ *is an isomorphism of A-modules.*

For our later application, let us present a proof here. The notation is the same as in Theorem 2.15.

Proof of Theorem 2.15: 1) Recall Proposition 2.4 shows that we have a natural morphism $P_{U_1} \to P_{U_2}$. Then we have an equivariant morphism $P_{U_1}^{\overset{L}{\boxtimes}\ell} \to P_{U_2}^{\overset{L}{\boxtimes}\ell}$. Applying F_ℓ, we obtain

$$F_\ell(U_1, \mathbb{K}) \xrightarrow{F_\ell(i,\mathbb{K})} F_\ell(U_2, \mathbb{K}). \tag{2.12}$$

Then the first part follows by taking stalks over T.

2) To prove the invariance we use the expression $C_T^{\mathbb{Z}/\ell}(U,\mathbb{K}) \cong \mathrm{RHom}_{\mathbb{Z}/\ell}(P_U^{\circledast\ell}, \beta_T'\mathbb{K}[-d])$ given by the adjoint isomorphism.

• It is shown in Corollary 2.5 that we have an isomorphism $P_{\varphi_z(U)} \cong \mathcal{K}^{-1} \star P_U \star \mathcal{K}$ where $\mathcal{K}$ is given using the GKS quantization of φ.

Let us write $\mathcal{K}_\ell = \mathcal{K}^{\circledast\ell}$, $\mathcal{K}_\ell^{-1} = (\mathcal{K}^{-1})^{\circledast\ell}$. We remark that $\mathcal{K}_\ell$ has a natural lift in the equivariant category and that $\mathcal{K}_\ell$, $\mathcal{K}_\ell^{-1}$ are mutually inverse for the convolution. Hence $\mathcal{K}_\ell \star -$ is an equivalence and $\mathrm{RHom}_{\mathbb{Z}/\ell}(G,H) \cong \mathrm{RHom}_{\mathbb{Z}/\ell}(\mathcal{K}_\ell \star G, \mathcal{K}_\ell \star H)$ for any $G, H \in D_{\mathbb{Z}/\ell}(X^{2l} \times \mathbb{R}_t)$. We denote by κ the auto-equivalence on $D_{\mathbb{Z}/\ell}(X^{2\ell} \times \mathbb{R}_t)$ induced by conjugation with $\mathcal{K}_\ell$:

$$\kappa(F) := \mathcal{K}_\ell^{-1} \star F \star \mathcal{K}_\ell. \tag{2.13}$$

Then we have an isomorphism $P_{\varphi_z(U)}^{\circledast\ell} \cong \mathcal{K}_\ell^{-1} \star P_U^{\circledast\ell} \star \mathcal{K}_\ell = \kappa(P_U^{\circledast\ell})$, and for $U = T^*X$, the isomorphism is realized by $\mathbb{K}_{\Delta_{X^2}\times[0,\infty)}^{\circledast\ell} \cong \mathcal{K}_\ell^{-1} \star \mathcal{K}_\ell \star \mathbb{K}_{\Delta_{X^2}\times[0,\infty)}^{\circledast\ell} \cong \mathcal{K}_\ell^{-1} \star \mathbb{K}_{\Delta_{X^2}\times[0,\infty)}^{\circledast\ell} \star \mathcal{K}_\ell$. Then the composition induces the isomorphism

$$\begin{aligned}\mathrm{RHom}_{\mathbb{Z}/\ell}(P_U^{\circledast\ell}, \beta_T'\mathbb{K}) &\overset{\kappa}{\cong} \mathrm{RHom}_{\mathbb{Z}/\ell}(\kappa(P_U^{\circledast\ell}), \kappa(\beta_T'\mathbb{K})) \\ &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}(P_{\varphi_z(U)}^{\circledast\ell}, \kappa(\beta_T'\mathbb{K})).\end{aligned}$$

• Therefore, to complete the proof, it is enough to construct an isomorphism $\kappa(\beta_T'\mathbb{K}) = \mathcal{K}_\ell^{-1} \star \beta_T'\mathbb{K} \star \mathcal{K}_\ell \cong \beta_T'\mathbb{K}$. Compared to Chiu's original proof, we will construct the isomorphism explicitly.

Notice that $\beta_T'\mathbb{K}$ is, up to orientation and shift, the constant sheaf on the graph of the permutation map $f\colon X^\ell \to X^\ell$, $(\mathbf{q}_1, \dots, \mathbf{q}_\ell) \mapsto (\mathbf{q}_2, \dots, \mathbf{q}_\ell, \mathbf{q}_1)$. Set $Y = X^\ell$ and identify $Y^2 = (X^2)^\ell$ by $(\mathbf{q}_1^1, \dots, \mathbf{q}_\ell^1, \mathbf{q}_1^2, \dots, \mathbf{q}_\ell^2) \mapsto$

$(\mathbf{q}_1^1, \mathbf{q}_1^2, \dots, \mathbf{q}_\ell^1, \mathbf{q}_\ell^2)$. Then, up to degree shifting, we have

$$\beta'_T\mathbb{K} \cong \mathbb{K}_{\Gamma_f\times\{T\}} \star E \cong E \star \mathbb{K}_{\Gamma_f\times\{T\}},$$

where $E = \delta_{Y^2!}(\omega_Y)\overset{L}{\boxtimes}\mathbb{K}_{\{0\}}$, with ω_Y the dualizing sheaf and δ_{Y^2} the usual diagonal embedding. In general, we have $E \star - \cong - \star E$.

Now we have the general fact $G \star \mathbb{K}_{\Gamma_g\times\{T\}} \cong (\mathrm{Id}_Y \times g \times \mathrm{T}_T)_!(G)$ for any G and any map g. This formula has the symmetric form $\mathbb{K}_{\Gamma'_g\times\{T\}} \star G \cong (g \times \mathrm{Id}_Y \times \mathrm{T}_T)_!(G)$ where Γ'_g is the switched graph $\Gamma'_g = \{(g(y), y) : y \in Y\}$. When g is invertible, we have $\Gamma_{g^{-1}} = \Gamma'_g$. So, we obtain

$$\mathcal{K}_\ell \star \beta'_T\mathbb{K} \cong \mathcal{K}_\ell \star \mathbb{K}_{\Gamma_f\times\{T\}} \star E \cong (\mathrm{Id}_Y \times f \times \mathrm{T}_T)_!(\mathcal{K}_\ell) \star E,$$

and

$$\begin{aligned}\beta'_T\mathbb{K} \star \mathcal{K}_\ell &\cong E \star \mathbb{K}_{\Gamma_f\times\{T\}} \star \mathcal{K}_\ell = E \star \mathbb{K}_{\Gamma'_{f^{-1}}\times\{T\}} \star \mathcal{K}_\ell \\ &\cong E \star (f^{-1} \times \mathrm{Id}_Y \times \mathrm{T}_T)_!(\mathcal{K}_\ell).\end{aligned}$$

In coordinate $(X^2)^\ell$ we have $(f \times f)((\mathbf{q}_j^1, \mathbf{q}_j^2))_{j\in\mathbb{Z}/\ell} = ((\mathbf{q}_{j+1}^1, \mathbf{q}_{j+1}^2))_{j\in\mathbb{Z}/\ell}$. In other words $f \times f$ is the cyclic permutation of the X^2 factors in $(X^2)^\ell$. It is then clear that $(f \times f \times \mathrm{Id}_\mathbb{R})_!\mathcal{K}_\ell \cong \mathcal{K}_\ell$ (even in the equivariant category). Then we deduced that

$$\begin{aligned}\beta'_T\mathbb{K} \star \mathcal{K}_\ell &\cong E \star (f^{-1} \times \mathrm{Id}_Y \times \mathrm{T}_T)_!(\mathcal{K}_\ell) \\ &\cong E \star (\mathrm{Id}_Y \times f \times \mathrm{T}_T)_!(\mathcal{K}_\ell) \cong \mathcal{K}_\ell \star \beta'_T\mathbb{K}.\end{aligned}$$

Consequently, we have

$$\kappa(\beta'_T\mathbb{K}) = \mathcal{K}_\ell^{-1} \star \beta'_T\mathbb{K} \star \mathcal{K}_\ell \cong \mathcal{K}_\ell^{-1} \star \mathcal{K}_\ell \star \beta'_T\mathbb{K} \cong \beta'_T\mathbb{K}.$$

In summary, the $\Phi_{z,T}^{\mathbb{Z}/\ell}$ is defined as following. For any $f \in \mathrm{Ext}^*_{\mathbb{Z}/\ell}(P_U^{\boxtimes\ell}, \beta'_T\mathbb{K}[-d])$, we have

$$\Phi_{z,T}^{\mathbb{Z}/\ell}(f) : P_{\varphi_z(U)}^{\boxtimes\ell} \cong \kappa(P_U^{\boxtimes\ell}) \xrightarrow{\kappa(f)} \kappa(\beta'\mathbb{K}[-d]) \cong \beta'\mathbb{K}[-d].$$

The functoriality of $\Phi_{z,T}^{\mathbb{Z}/\ell}$ follows since κ is a functor.

For $U = T^*X$, the isomorphism $P_{T^*X}^{\boxtimes\ell} \cong \kappa(P_{T^*X}^{\boxtimes\ell})$ is induced by the natural isomorphism $P_{T^*X} \star \mathcal{K} \cong \mathcal{K} \star P_{T^*X}$. So does $\kappa(\beta'\mathbb{K}[-d]) \cong \beta'\mathbb{K}[-d]$. Then the induced isomorphism $\Phi_{z,T}^{\mathbb{Z}/\ell}(f)$ is the identity on the cohomology level. □

Actually, the isomorphism $\kappa(\beta'_T\mathbb{K})\cong\beta'_T\mathbb{K}$ is still true if we replace $\mathbb{K}$ by $M \in D_{\mathbb{Z}/\ell}(\mathrm{pt})$, and moreover the isomorphism is functorial with respect to M. In fact, for $M \in D_{\mathbb{Z}/\ell}(\mathrm{pt})$, we only need to replace $K = \delta_{Y^2!}(\pi_Y^!\mathbb{K})$ in the proof by $K(M) = \delta_{Y^2!}(\pi_Y^! M)$. Consequently, we can construct an isomorphism of functors

$$\Phi_{z,T}^{\mathbb{Z}/\ell}(-) : \mathrm{RHom}_{\mathbb{Z}/\ell}(F_\ell(U,\mathbb{K})_T, -) \xrightarrow{\cong} \mathrm{RHom}_{\mathbb{Z}/\ell}(F_\ell(\varphi(U),\mathbb{K})_T, -).$$

Now, let us take $M = F_\ell(\varphi(U),\mathbb{K})_T$. Then $\mathrm{Id}_{F_\ell(\varphi(U),\mathbb{K})_T}$ provide us with an isomorphism $\Phi_{z,T}^{\mathbb{Z}/\ell'}$ defined as

$$\left(\Phi_{z,T}^{\mathbb{Z}/\ell}(F_\ell(\varphi(U),\mathbb{K})_T)\right)^{-1}(\mathrm{Id}_{F_\ell(\varphi(U),\mathbb{K})_T}) : F_\ell(U,\mathbb{K})_T \xrightarrow{\cong} F_\ell(\varphi(U),\mathbb{K})_T.$$

In summary, we have

Proposition 2.17. *For a compactly supported Hamiltonian isotopy $\varphi : I \times T^*X \to T^*X$, there exists an isomorphism, in the equivariant category, $\Phi_{z,T}^{\mathbb{Z}/\ell'} : F_\ell(U,\mathbb{K})_T \to F_\ell(\varphi_z(U),\mathbb{K})_T$, for all $z \in I$.*

Remark 2.18. In [Zha23, Subsection 3.7], we explain how to give a new proof of the Proposition 2.17 without using adjunctions. We also explain that the proposition is true for Hamiltonian homeomorphism in loc. cit.

2.3. Geometry of $F_\ell(U,\mathbb{K})$

In this subsection, we assume that U is dynamically admissible (Definition 2.9) and we give a more accessible expression of the Chiu-Tamarkin complex using sheaf quantization. We then discuss the underlying geometry.

Following ideas of Chiu, we first compute $F_\ell(U,\mathbb{K}) \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\widetilde{\Delta}_X^{-1}\left(\mathrm{R}s_{t!}^{\ell}P_U^{\overset{L}{\boxtimes}\ell}\right)$ going back to the construction of P_U.

We recall that $\mathcal{K}$ is the sheaf quantization of a Hamiltonian $\mathbb{R}_z^m$ action on T^*X with a moment map μ, $\Omega \subset \mathbb{R}_\zeta^m$ and $U = \mu^{-1}(\Omega)$. Then we have $P_U \cong \widehat{\mathcal{K}} \circ \mathbb{K}_\Omega \cong \mathcal{K} \star \widehat{\mathbb{K}_\Omega}$.

As a corollary of the proper base change and the projection formula, we have the following:

$$P_U^{\overset{L}{\boxtimes}\ell} \cong (\mathcal{K} \star \widehat{\mathbb{K}_\Omega})^{\overset{L}{\boxtimes}\ell} \cong \mathrm{R}\pi_{\underline{z}!}\mathrm{R}s^2_{\mathbb{R}^\ell!}\left(\pi_{t_2}^{-1}\mathcal{K}^{\overset{L}{\boxtimes}\ell}\overset{L}{\otimes}\pi_{t_1}^{-1}\widehat{\mathbb{K}_\Omega}^{\overset{L}{\boxtimes}\ell}\right).$$

Next, we have

$$\begin{aligned} F_\ell(U,\mathbb{K}) &\cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\widetilde{\Delta}_X^{-1}\mathrm{R}s^\ell_{t!}\mathrm{R}\pi_{\underline{z}!}\mathrm{R}s^2_{\mathbb{R}^\ell!}\left(\pi_{t_2}^{-1}\mathcal{K}^{\overset{L}{\boxtimes}\ell}\overset{L}{\otimes}\pi_{t_1}^{-1}\widehat{\mathbb{K}_\Omega}^{\overset{L}{\boxtimes}\ell}\right)\\ &\cong \mathrm{R}\pi_{\underline{z}!}\mathrm{R}s^\ell_{t!}\mathrm{R}s^2_{\mathbb{R}^\ell!}\left(\pi_{t_2}^{-1}\left(\mathrm{R}\pi_{\underline{\mathbf{q}}!}\widetilde{\Delta}_X^{-1}\mathcal{K}^{\overset{L}{\boxtimes}\ell}\right)\overset{L}{\otimes}\pi_{t_1}^{-1}\widehat{\mathbb{K}_\Omega}^{\overset{L}{\boxtimes}\ell}\right), \end{aligned}$$

where $\underline{z}=(z_1,\dots,z_\ell)\in(\mathbb{R}^m)^\ell$, $t_i=(t_i^1,\dots,t_i^\ell)\in\mathbb{R}^\ell$ for $i=1,2$, and $t=(t^1,\dots,t^\ell)=s^2_{\mathbb{R}^\ell}(t_1,t_2)$. Now, let $z=z_1+\cdots+z_\ell$ and take $t_i'=t_i^1+\cdots+t_i^\ell$. Using this change of coordinate, we have the decomposition $\pi_{\underline{z}}=\pi_z s_z^\ell$ and $s_t^\ell s^2_{\mathbb{R}^\ell}=s^2_{t'}(s^\ell_{t_1}\times s^\ell_{t_2})$. Therefore, we obtain

$$\begin{aligned} &F_\ell(U,\mathbb{K})\\ \cong&\mathrm{R}\pi_{\underline{z}!}\mathrm{R}s^\ell_{t!}\mathrm{R}s^2_{\mathbb{R}^\ell!}\left(\pi_{t_2}^{-1}\left(\mathrm{R}\pi_{\underline{\mathbf{q}}!}\widetilde{\Delta}^{-1}\mathcal{K}^{\overset{L}{\boxtimes}\ell}\right)\overset{L}{\otimes}\pi_{t_1}^{-1}\widehat{\mathbb{K}_\Omega}^{\overset{L}{\boxtimes}\ell}\right)\\ \cong&\mathrm{R}\pi_{z!}\mathrm{R}s^2_{t'!}\mathrm{R}s^\ell_{z!}\mathrm{R}(s^\ell_{t_1}\times s^\ell_{t_2})_!\left(\pi_{t_2}^{-1}\left(\mathrm{R}\pi_{\underline{\mathbf{q}}!}\widetilde{\Delta}^{-1}\mathcal{K}^{\overset{L}{\boxtimes}\ell}\right)\overset{L}{\otimes}\pi_{t_1}^{-1}\widehat{\mathbb{K}_\Omega}^{\overset{L}{\boxtimes}\ell}\right)\\ \cong&\mathrm{R}\pi_{z!}\mathrm{R}s^2_{t'!}\mathrm{R}s^\ell_{z!}\left(\pi_{t_2'}^{-1}\left(\mathrm{R}\pi_{\underline{\mathbf{q}}!}\widetilde{\Delta}^{-1}\mathcal{K}^{\boxstar\ell}\right)\overset{L}{\otimes}\pi_{t_1'}^{-1}\widehat{\mathbb{K}_\Omega}^{\boxstar\ell}\right). \end{aligned} \tag{2.14}$$

This formula shows, as the construction itself, that we can consider separately the Hamiltonian action and the cut-off by Ω. Let us study the Hamiltonian action first. In view of (2.14), it is convenient to define

$$\begin{aligned} CL_\ell(\mathcal{K})&\coloneqq\mathrm{R}\pi_{\underline{\mathbf{q}}!}(\widetilde{\Delta}^{-1}(\mathcal{K}^{\boxstar\ell}))\in D_{\mathbb{Z}/\ell}((\mathbb{R}^m_z)^\ell\times\mathbb{R}_t),\\ \mathcal{CL}_\ell(\mathcal{K})&\coloneqq\mathrm{R}s^\ell_{z*}CL_\ell(\mathcal{K})\in D_{\mathbb{Z}/\ell}(\mathbb{R}^m_z\times\mathbb{R}_t). \end{aligned} \tag{2.15}$$

Noticed that the formula $CL_\ell(\mathcal{K})=\mathrm{R}\pi_{\underline{\mathbf{q}}!}(\widetilde{\Delta}^{-1}(\mathcal{K}^{\boxstar\ell}))$ is similar to $F'_\ell(P_U^{\boxstar\ell})=\mathrm{R}\pi_{\underline{\mathbf{q}}!}(\widetilde{\Delta}^{-1}(P_U^{\boxstar\ell}))$ (see (2.8)). But in these two formulas, $\pi_{\underline{\mathbf{q}}}$ has different meaning. The codomain of the first $\pi_{\underline{\mathbf{q}}}$ is $\mathbb{R}^m_z\times\mathbb{R}_t$ while the codomain of second $\pi_{\underline{\mathbf{q}}}$ is just $\mathbb{R}_t$. So they are different formulas.

The sheaves $CL_\ell(\mathcal{K})$ and $\mathcal{CL}_\ell(\mathcal{K})$ encode the cohomology information of a discrete Hamiltonian loop space. Precisely, we have

Proposition 2.19. *With the notation* (2.15) *we have*

1) The sectional microsupport $\mu s(CL_\ell(\mathcal{K}))$, which is a subset of $T^*(\mathbb{R}^m_z)^\ell$, is contained in

$$\left\{(z_j,\zeta_j)_{j\in\mathbb{Z}/\ell} : \begin{array}{l}\textit{There exist } (\boldsymbol{q}_j,\boldsymbol{p}_j)_{j\in\mathbb{Z}/\ell}\in T^*((X^2)^\ell) \textit{ such that}\\ (\boldsymbol{q}_{j+1},\boldsymbol{p}_{j+1}) = z_j\cdot(\boldsymbol{q}_j,\boldsymbol{p}_j),\ \zeta_j=-\mu(\boldsymbol{q}_j,\boldsymbol{p}_j)\,,j\in\mathbb{Z}/\ell\end{array}\right\}$$

2) $CL_\ell(\mathcal{K})\cong (s^\ell_z)^{-1}\mathrm{R}s^\ell_{z*}CL_\ell(\mathcal{K})$, $\mathcal{CL}_\ell(\mathcal{K})\cong \mathrm{R}s^\ell_{z*}(s^\ell_z)^{-1}\mathcal{CL}_\ell(\mathcal{K})$.

Proof. 1) It follows directly from the functorial estimate of microsupport. First, the formula (1.10) shows that

$$\mu s(\mathcal{K}^{\boxtimes\ell})\subset\{(z_j,\zeta_j,\mathbf{q}_j,-\mathbf{p}_j,\mathbf{q}'_j,\mathbf{p}'_j)_{j\in\mathbb{Z}/\ell} : (\mathbf{q}'_j,\mathbf{p}'_j)=z_j\cdot(\mathbf{q}_j,\mathbf{p}_j),\ j\in\mathbb{Z}/\ell\}.$$

The transpose derivative of $\widetilde{\Delta}$ is given by

$$\begin{aligned}&d\widetilde{\Delta}^*(\mathbf{q}_\ell,\mathbf{q}_1,\ldots,\mathbf{q}_{\ell-1},\mathbf{q}_\ell;\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_{2\ell-1},\mathbf{p}_{2\ell})\\&\quad=(\mathbf{q}_1,\ldots,\mathbf{q}_\ell;\mathbf{p}_2+\mathbf{p}_3,\ldots,\mathbf{p}_{2\ell}+\mathbf{p}_1).\end{aligned}$$

By the bound Theorem 1.4-(3), we deduce that $\mu s(\widetilde{\Delta}^{-1}(\mathcal{K}^{\boxtimes\ell}))$ is a subset of

$$\left\{(z_j,\zeta_j,\mathbf{q}''_j,\mathbf{p}''_j)_{j\in\mathbb{Z}/\ell} : \begin{array}{l}\text{There exist } (\mathbf{q}_j,-\mathbf{p}_j,\mathbf{q}'_j,\mathbf{p}'_j)_{j\in\mathbb{Z}/\ell}\in T^*((X^2)^{2\ell})\\ \text{ such that } (\mathbf{q}'_j,\mathbf{p}'_j)=z_j\cdot(\mathbf{q}_j,\mathbf{p}_j),\\ \mathbf{q}''_j=\mathbf{q}'_j=\mathbf{q}_{j+1},\ \mathbf{p}''_j=\mathbf{p}'_j-\mathbf{p}_{j+1},\zeta_j=-\mu(\mathbf{q}_j,\mathbf{p}_j),\\ j\in\mathbb{Z}/\ell\end{array}\right\}.$$

Finally, let us apply the non proper estimate Theorem 1.7. The set $\pi^{\#}_{\underline{\mathbf{q}''}}(SS(\widetilde{\Delta}^{-1}(\mathcal{K}^{\boxtimes\ell})))$ comes from forgetting $\mathbf{q}''_j$ for all j from $SS(\widetilde{\Delta}^{-1}(\mathcal{K}^{\boxtimes\ell}))$. Then $(z_j,\zeta_j,t,1)_{j\in\mathbb{Z}/\ell}\in\mu s(CL_\ell(\mathcal{K}))$ if there exists a sequence $(z^n_j,\zeta^n_j,\mathbf{p}''^n_j)_{j\in\mathbb{Z}/\ell}\in\pi^{\#}_{\underline{\mathbf{q}''}}(SS(\widetilde{\Delta}^{-1}(\mathcal{K}^{\boxtimes\ell})))$ such that $z^n_j\to z_j$, $\zeta^n_j\to\zeta_j$, and $\mathbf{p}''^n_j\to 0$ for all $j\in\mathbb{Z}/\ell$.

On the other hand, the relations above imply that there exists $(\mathbf{q}^n_j,-\mathbf{p}^n_j,{\mathbf{q}_j}'^n,{\mathbf{p}_j}'^n)_{j\in\mathbb{Z}/\ell}\in T^*((X^2)^\ell)$ such that $({\mathbf{q}_j}'^n,{\mathbf{p}_j}'^n)=z^n_j\cdot(\mathbf{q}^n_j,\mathbf{p}^n_j)$ and $\mathbf{q}''^n_j=\mathbf{q}'^n_j=\mathbf{q}^n_{j+1}$, $\mathbf{p}''^n_j=\mathbf{p}'^n_j-\mathbf{p}^n_{j+1},\zeta^n_j=-\mu(\mathbf{q}^n_j,\mathbf{p}^n_j)$ for all $j\in\mathbb{Z}/\ell$. So the continuity of the group action and the moment map show that, after taking limit $n\to\infty$, we have $(\mathbf{q}'_j,\mathbf{p}'_j)=z_j\cdot(\mathbf{q}_j,\mathbf{p}_j)$ and $\mathbf{q}''_j=\mathbf{q}'_j=\mathbf{q}_{j+1}$, $0=\mathbf{p}'_j-\mathbf{p}_{j+1},\zeta_j=-\mu(\mathbf{q}_j,\mathbf{p}_j)$ for all $j\in\mathbb{Z}/\ell$. Then we have that

$\mu s(CL_\ell(\mathcal{K}))$ is contained in

$$\left\{ (z_j,\zeta_j)_{j\in\mathbb{Z}/\ell} : \begin{array}{l} \text{There exist } (\mathbf{q}_j, -\mathbf{p}_j, \mathbf{q}'_j, \mathbf{p}'_j)_{j\in\mathbb{Z}/\ell} \in T^*((X^2)^\ell) \\ \text{such that } (\mathbf{q}'_j, \mathbf{p}'_j) = z_j \cdot (\mathbf{q}_j, \mathbf{p}_j), \\ \mathbf{q}''_j = \mathbf{q}'_j = \mathbf{q}_{j+1},\ 0 = \mathbf{p}''_j = \mathbf{p}'_j - \mathbf{p}_{j+1}, \zeta_j = -\mu(\mathbf{q}_j, \mathbf{p}_j)\,, \\ j \in \mathbb{Z}/\ell \end{array} \right\}.$$

Finally, we simplify the notation by reducing the variables with primes.

2) If $(z_j,\zeta_j)_{j\in\mathbb{Z}/\ell} \in \mu s(CL_\ell(\mathcal{K}))$, there exists $(\mathbf{q}_j, \mathbf{p}_j)_{j\in\mathbb{Z}/\ell} \in T^*((X^2)^\ell)$ such that $(\mathbf{q}_{j+1}, \mathbf{p}_{j+1}) = z_j \cdot (\mathbf{q}_j, \mathbf{p}_j)$ for all $j \in \mathbb{Z}/\ell$. Therefore, the invariance of the moment map shows that

$$\zeta_{j+1} = \mu(\mathbf{q}_{j+1}, \mathbf{p}_{j+1}) = (\mathbf{q}_j, \mathbf{p}_j) = \zeta_j, \qquad j \in \mathbb{Z}/\ell.$$

Then, the isomorphism follows from [KS90, Proposition 5.4.5(ii)]. □

Remark 2.20. Even if $\mathcal{K}$ is a sheaf quantization that comes from a non-autonomous Hamiltonian function, the microsupport estimate for $CL_\ell(\mathcal{K})$ is still true, but the second statement is not true in this case.

Now, using the projection formula, we can write the formula (2.14) as

$$F_\ell(U,\mathbb{K}) \cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\left(\pi_{t_2}^{-1}\mathcal{CL}_\ell(\mathcal{K})\overset{L}{\otimes}\pi_{t_1}^{-1}\mathrm{R}s^\ell_{z!}\widehat{\mathbb{K}_\Omega}^{\circledast\ell}\right). \tag{2.16}$$

Next, let us study $\mathrm{R}s^\ell_{z!}\widehat{\mathbb{K}_\Omega}^{\circledast\ell} \cong \mathrm{R}s^\ell_{(z,t_2)!}\widehat{\mathbb{K}_\Omega}^{\overset{L}{\boxtimes}\ell}$. First, with the help of [D'A13, Section 6, Appendix A], $\widehat{\mathbb{K}_\Omega}$ is the (inverse) Fourier-Sato transform $\widehat{\mathbb{K}_{\Omega'}}$ of $\mathbb{K}_{\Omega'}$, where $\Omega' = \{(\zeta,\tau) : \tau\zeta \in \Omega, \tau > 0\}$. Now, using the functorial properties of the Fourier-Sato transformation (see [KS90, Section 3.7]), and writing in the same way the two Fourier transforms, we have:

$$\mathrm{R}s^\ell_{z!}\widehat{\mathbb{K}_\Omega}^{\circledast\ell} \cong \mathrm{R}s^\ell_{(z,t_2)!}\widehat{\mathbb{K}_{\Omega'}}^{\overset{L}{\boxtimes}\ell} \cong \mathrm{R}s^\ell_{(z,t_2)!}\widehat{\mathbb{K}_{\Omega'}^{\overset{L}{\boxtimes}\ell}} \cong \mathrm{R}s^\ell_{(z,t_2)!}\widehat{\mathbb{K}_{\Omega'^\ell}} \cong \widehat{({}^ts^\ell_{(z,t_2)})^{-1}\mathbb{K}_{\Omega'^\ell}}.$$

Since the transpose of the summation map $s^\ell_{(z,t_2)}$ is the diagonal map $\delta_{(z,t_2)^\ell}$, we conclude that

$$\mathrm{R}s^\ell_{z!}\widehat{\mathbb{K}_\Omega}^{\circledast\ell} \cong \mathrm{R}s^\ell_{(z,t_2)!}\widehat{\mathbb{K}_{\Omega'}}^{\overset{L}{\boxtimes}\ell} \cong \widehat{\delta^{-1}_{(z,t_2)^\ell}\mathbb{K}_{\Omega'^\ell}} \cong \widehat{\mathbb{K}_{\Omega'}} \cong \widehat{\mathbb{K}_\Omega}.$$

Our external tensor power is in fact an object of the $\mathbb{Z}/\ell$-equivariant derived category. We need to mention that the Fourier transform (of any version) is a convolution functor defined by a kernel, which is a constant sheaf supported on a closed subset. So, on the product space, the Fourier transform is defined by a kernel that is a constant sheaf supported on a *product* of the same closed subsets. Then the kernel is a $\mathbb{Z}/\ell$-equivariant sheaf. Moreover, the external tensor power is compatible with the Grothendieck 6-operations. Therefore, the Fourier transform can be defined on the equivariant derived category. Finally, all maps here are $\mathbb{Z}/\ell$-equivariant with respect to cyclic permutation action and the formulas we used here are valid in the equivariant category. In conclusion, all identities here are true in the equivariant derived category.

Consequently, (2.16) could be read as

(2.17)
$$F_\ell(U,\mathbb{K}) \cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\left(\pi_{t_2}^{-1}\mathcal{CL}_\ell(\mathcal{K})\overset{L}{\otimes}\pi^{-1}_{(\underline{\mathbf{q}},t_1)}\widehat{\mathbb{K}_\Omega}\right) \cong \mathrm{R}\pi_{z!}\left(\mathcal{CL}_\ell(\mathcal{K})\star\widehat{\mathbb{K}_\Omega}\right).$$

From this formula, the study of $F_\ell(U,\mathbb{K})$ is reduced to understanding $\mathcal{CL}_\ell(\mathcal{K})$.

The case $m=1$ is particularly useful for our applications. Now $\Omega = (-\infty,1)$ and $\widehat{\mathbb{K}_\Omega} \cong \mathbb{K}_{\{(z,t):-t\leq z\leq 0\}}$. For $T\geq 0$, (2.17) shows

(2.18)
$$\alpha_{\ell,X,T}(P_U^{\overset{L}{\boxtimes}\ell}) \cong F_\ell(U,\mathbb{K})_T \cong \mathrm{R}\Gamma_c\left(\mathbb{R}_z\times\mathbb{R}^2_{(t_1,t_2)};\left(\mathcal{CL}_\ell(\mathcal{K})\overset{L}{\boxtimes}\mathbb{K}_{\mathbb{R}_{t_2}}\right)_Z\right),$$

where $Z=\{(z,t_1,t_2): t_1+t_2=T, -t_2\leq z\leq 0\}$.

Again, using the formula (2.17), we obtain the following action spectrum estimate of the microsupport of $F_\ell(U,\mathbb{K})$ for dynamically admissible sets.

Lemma 2.21. *[Zha20, formula 74]Let $U=\{H<1\}$ be a dynamically admissible set defined by a Hamiltonian function H. If the boundary ∂U is a non-degenerated hypersurface of restricted contact type (RCT) given by $\partial U=\{H=1\}$, then we have*
(2.19)
$$\mu s_L(F_\ell(U,\mathbb{K}))\subset\left\{t\in\mathbb{R}: t=\left|\int_c \boldsymbol{p}d\boldsymbol{q}\right| \text{ for a closed orbit } c \text{ of } \varphi_z^H \text{ in } \partial U\right\}.$$

Actually, since $F_1(U,\mathbb{K})\cong F_\ell(U,\mathbb{K})$ in $D(\mathbb{R})$ (by (2.9)), we only need to verify the estimate for $F_1(U,\mathbb{K})$ (see Definition 1.21). Notice that when computing the upper bound, we need the contact boundary condition to make sure we can attach only one non-constant closed characteristic.

Geometrically, we call the right hand side *the action spectrum* of the Reeb action in ∂U.

So far, we have found two different ways to understand $F_\ell(U,\mathbb{K})$. Initially, from the definition of $F_\ell(U,\mathbb{K})$, we first cut off the energy of a Hamiltonian isotopy up to Legendre transform to obtain the kernels and then use the functor α_T to obtain cohomology of some discrete loop space with action bound T. On the other hand, we can study discrete loops of a Hamiltonian isotopy first, and then cut off energy up to Legendre transform. The result of the section clarifies that these two ways are the same. The second way is more direct than the first in many cases; we will see more about this point of view when doing computation for toric domains.

2.4. Fundamental class and capacities

Now, let us assume that X is an oriented manifold of dimension d with a fixed orientation and $\mathbb{K}$ is a field. For an admissible open subset $U \xhookrightarrow{i_U} T^*X$ and $T \geq 0$, Theorem 2.15-(1) shows that we have a morphism in the $\mathbb{Z}/\ell$-equivariant derived category:

$$C_T^{\mathbb{Z}/\ell}(T^*X,\mathbb{K}) \xrightarrow{i_U^*} C_T^{\mathbb{Z}/\ell}(U,\mathbb{K}),$$

and it induces a morphism of $A = \mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{K},\mathbb{K})$-module on cohomology

$$H^{BM}_{d-*}(X,\mathbb{K}) \otimes A \cong H^*C_T^{\mathbb{Z}/\ell}(T^*X,\mathbb{K}) \xrightarrow{i_U^*} H^*C_T^{\mathbb{Z}/\ell}(U,\mathbb{K}),$$

where the first isomorphism is given in (2.10). Since X is orientable, we have the fundamental class $[X]$ of X in $H_d^{BM}(X,\mathbb{K})$, which is defined via $1 \in H^0(X,\mathbb{K}) \cong H_d^{BM}(X,\mathbb{K})$. We set $[X]^{\mathbb{Z}/\ell} = [X] \otimes 1$, where $1 \in A$ is the identity.

Definition 2.22. For an admissible open set $U \xhookrightarrow{i_U} T^*X$, and $T \geq 0$, we define its *fundamental class* $\eta_T^{\mathbb{Z}/\ell}(U,\mathbb{K})$ as the image of $[X]^{\mathbb{Z}/\ell}$ under i_U^*. That is, $\eta_T^{\mathbb{Z}/\ell}(U,\mathbb{K}) := i_U^*([X]^{\mathbb{Z}/\ell}) \in H^0C_T^{\mathbb{Z}/\ell}(U,\mathbb{K})$. When $\ell = 1$, we use $\eta_T(U,\mathbb{K})$ for short.

By definition, the fundamental class can be computed as the following composition:

$$\begin{aligned}(F_\ell(U,\mathbb{K}))_T &\to (F_\ell(T^*X,\mathbb{K}))_T \cong \mathrm{R}\Gamma_c(X,\mathbb{K}) \\ &\xrightarrow{or} H^d\mathrm{R}\Gamma_c(X,\mathbb{K})[-d] \cong \mathbb{K}[-d].\end{aligned} \tag{2.20}$$

As a corollary of Theorem 2.15, we have

Proposition 2.23. *1) Let $U \subset U' \subset T^*X$ be an inclusion of admissible open sets. Through the natural morphism*

$$H^0 C_T^{\mathbb{Z}/\ell}(U', \mathbb{K}) \to H^0 C_T^{\mathbb{Z}/\ell}(U, \mathbb{K})$$

we have

$$\eta_T^{\mathbb{Z}/\ell}(U', \mathbb{K}) \mapsto \eta_T^{\mathbb{Z}/\ell}(U, \mathbb{K}).$$

2) Let $\varphi : I \times T^*X \to T^*X$ be a compactly supported Hamiltonian isotopy and U be an admissible open set. Recall the A-module isomorphism, defined in Theorem 2.15,

$$H^*(\Phi_{z,T}^{\mathbb{Z}/\ell}) : H^* C_T^{\mathbb{Z}/\ell}(U, \mathbb{K}) \xrightarrow{\cong} H^* C_T^{\mathbb{Z}/\ell}(\varphi_z(U), \mathbb{K}).$$

Then we have $H^0(\Phi_{z,T}^{\mathbb{Z}/\ell})(\eta_T^{\mathbb{Z}/\ell}(U, \mathbb{K})) = \eta_T^{\mathbb{Z}/\ell}(\varphi_z(U), \mathbb{K})$ for all $z \in I$.

We have $\eta_T^{\mathbb{Z}/\ell}(T^*X, \mathbb{K}) = [X]^{\mathbb{Z}/\ell}$ for all $T \geq 0$. So, if there exists an open set $X' \subset X$ such that $U \subset T^*X' \subset T^*X$, we have $\eta_T^{\mathbb{Z}/\ell}(U, \mathbb{K}) = i_U^*([X]^{\mathbb{Z}/\ell}) = i_U^*([X']^{\mathbb{Z}/\ell})$ by Proposition 2.23-(1).

Now, for $\ell \in \mathbb{N}_{\geq 2}$, p_ℓ is the minimal prime factor of ℓ, and $\mathbb{F}_{p_\ell}$ is the finite field of order p_ℓ. The Yoneda algebra $A = \operatorname{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{F}_{p_\ell}, \mathbb{F}_{p_\ell})$ is isomorphic to $\mathbb{F}_{p_\ell}[u, \theta]$ (see (1.15)), where $|u| = 2$, $|\theta| = 1$, and $\theta^2 = ku$ ($k = 0$ if ℓ is odd and $k = \ell/2$ if ℓ is even).

Definition 2.24. For an admissible open set U and $k \in \mathbb{N}$ we define

$$\operatorname{Spec}(U, k) := \left\{ T \geq 0 : \begin{array}{l} \exists p \text{ prime such that } \forall \ell \in \mathbb{N}_{\geq 2},\, p_\ell \geq p, \\ \eta_T^{\mathbb{Z}/\ell}(U, \mathbb{F}_{p_\ell}) \in u^k H^* C_T^{\mathbb{Z}/\ell}(U, \mathbb{F}_{p_\ell}) \end{array} \right\},$$

and

$$c_k(U) := \inf \operatorname{Spec}(U, k) \in [0, +\infty]. \tag{2.21}$$

For a general open set U, we define

$$c_k(U) = \sup\{c_k(U') : U' \subset U,\, U' \text{ is admissible}\}.$$

In the following, we will prove that $(c_k)_{k \in \mathbb{N}}$ defines a sequence of non-trivial symplectic capacities.

Theorem 2.25. *The functions $c_k : Open(T^*X) \to [0, \infty]$ satisfy the following:*

1) $c_k \leq c_{k+1}$ *for all* $k \in \mathbb{N}$.

2) For two open sets $U_1 \subset U_2$, *we have* $c_k(U_1) \leq c_k(U_2)$.

3) For a compactly supported Hamiltonian isotopy $\varphi : I \times T^*X \to T^*X$, *we have* $c_k(U) = c_k(\varphi_z(U))$.

4) If $X = \mathbb{R}^d$, *then* $c_k(rU) = r^2 c_k(U)$ *for all* $k \in \mathbb{N}$ *and* $r > 0$.

5) Suppose $U = \{H < 1\}$ *is admissible such that* $\partial U = \{H = 1\}$ *is a non-degenerated hypersurface of restricted contact type defined by a Hamiltonian function* H. *If* $c_k(U) < \infty$, *then* $c_k(U)$ *is represented by the action of a closed characteristic in the boundary* ∂U.

6) $c_k(U) > 0$ *for all open sets* U.

Proof. We can assume U is admissible; the general case follows directly. Then (1) is a consequence of Definition 2.24. Results (2), (3) are corollaries of Proposition 2.23.

For (4), recall that Corollary 2.6 shows that $P_{rU} \cong \bar{R}_! P_U$ where $\bar{R}(\mathbf{q}, \mathbf{q}', t) = (\mathbf{q}/r, \mathbf{q}'/r, t/r^2)$. Then direct computation shows that we have an isomorphism in $D_{\mathbb{Z}/\ell}(\mathbb{R})$:

$$R_! F_\ell(rU, \mathbb{K}) \cong F_\ell(U, \mathbb{K}),$$

where $R(t) = t/r^2$. In particular, we have $F_\ell(rU, \mathbb{K})_{r^2T} \cong F_\ell(U, \mathbb{K})_T$ for $T \geq 0$. This isomorphism commutes with the inclusion morphism induced by $U \subset T^*\mathbb{R}^d$, the (4) follows.

For (5), let $T = c_k(U)$. Suppose that it is not given by the action of a closed characteristic.

By assumption, the boundary ∂U has non-degenerated Reeb dynamics, so there are only finitely many closed characteristics with action less than $2T$. So there is a small $\varepsilon > 0$ such that there is no action happening in $[T - \varepsilon, T + \varepsilon]$.

However, we have the following microsupport estimate (2.19) for all fields $\mathbb{K}$:

$$\mu s_L(F_\ell(U, \mathbb{K})) \subset \left\{ t \in \mathbb{R} : t = \left| \int_c \mathbf{p} d\mathbf{q} \right| \text{ for some closed orbit } c \text{ of } \varphi_z^H \right\}.$$

Therefore $F_\ell(U, \mathbb{K})$ is constant on $[T - \varepsilon, T + \varepsilon]$. Consequently, $(F_\ell(U, \mathbb{K}))_{T-\varepsilon} \cong (F_\ell(U, \mathbb{K}))_T$, and then $\eta^{\mathbb{Z}/\ell}_{T-\varepsilon}(U, \mathbb{K}) = \eta^{\mathbb{Z}/\ell}_T(U, \mathbb{K})$ for all ℓ and all $\mathbb{K}$, in particular for $\mathbb{K} = \mathbb{F}_{p_\ell}$ for all $\ell \in \mathbb{N}$. Then we have

$c_k(U) \leq T - \varepsilon$, which gives a contradiction. So we have

$$c_k(U) \in \left\{ \left| \int_c \mathbf{p} d\mathbf{q} \right| : c \text{ is a closed orbit of } \varphi_z^H \right\}. \tag{2.22}$$

Finally, let us prove that the c_k's are positive. We will see, in Corollary 3.8, that for a ball B_a, one has $c_k(B_a) = \lceil k/d \rceil a$.

For a general admissible open set U, we can assume that there exists $(\mathbf{q}, 0) \in U$ by applying a compactly support cut-off of a translation along p-direction, which is a Hamiltonian map. It does not change $c_k(U)$ by (3). Then we take a neighborhood $X' \cong \mathbb{R}^d$ of $\mathbf{q}$. By (2), we have $c_k(U) \geq c_k(U \cap T^*X')$. To prove $c_k(U \cap T^*X') > 0$, let us take an admissible open subset W of $U \cap T^*X'$ such that $(\mathbf{q}, 0) \in W$.

On the other hand, the functorial property Proposition 2.23-(1) shows that $\eta_T^{\mathbb{Z}/\ell}(W, \mathbb{K}) = i_W^*([X]^{\mathbb{Z}/\ell}) = i_W^*([X']^{\mathbb{Z}/\ell})$. So, we only need to think W as an open subset of $T^*X' \cong T^*\mathbb{R}^d$, and then we can assume $X = X' = \mathbb{R}^d$ and $\mathbf{q} = 0$ now. We take a standard symplectic ball $B_a \subset W$, then $c_k(W) \geq c_k(B_a) > 0$. Consequently, we have $c_k(U) \geq c_k(U \cap T^*X') \geq c_k(W) > 0$. $\square$

Remark 2.26. We also see from $c_k(B_a) = \lceil k/d \rceil a$ that if U is a bounded open set (which is admissible by Corollary 2.10), then $c_k(U) < \infty$.

Remark 2.27. Finally, let us remark about the computability of c_k. As $H^*C_T^{\mathbb{Z}/\ell}(U, \mathbb{K})$ is defined using P_U, which is an object in the derived category. Although it is unique in the derived category, we can take different chain representatives of P_U. Therefore, to compute $c_k(U)$, we can choose a particular chain representative of P_U. Usually, these chain representatives of P_U admit properties that are not so obvious from general existence results like Proposition 2.8, and Corollary 2.10.

In Section 3, we will see how to construct a chain representative of P_{X_Ω}, for a toric domain X_Ω, using generating functions. The particular chain representative helps us to compute capacities for convex toric domains.

3. Toric domains

The 2-dimensional rotation $\varphi_z(u) = \exp(-2i\pi z)u$ on $\mathbb{C}_u$ is the Hamiltonian flow of the Hamiltonian function $H(u) = \pi|u|^2$. Here, we identify $\mathbb{C}_u$ with $T^*\mathbb{R}_q$ by $u = q + ip$.

Consider the product action of single 2-dimensional rotations given by

$$z \cdot (u_1, \ldots, u_n) = (\exp(-2i\pi z_1)u_1, \ldots, \exp(-2i\pi z_d)u_d).$$

This is a Hamiltonian action of $\mathbb{R}^d_z$, which is indeed a torus action, on $\mathbb{C}^d_u = T^*V$, where $V = \mathbb{R}^d_{\mathbf{q}}$ is a real vector space of dimension d, and $u = \mathbf{q} + i\mathbf{p}$. We call it the standard Hamiltonian torus action on $\mathbb{C}^d_u = T^*V$.

The moment map of the standard Hamiltonian torus action is

$$\mu : \mathbb{C}^d_u = T^*V \to (\mathbb{R}^d_z)^* = \mathbb{R}^d_\zeta, \quad \mu(u_1, \dots, u_n) = (\pi|u_1|^2, \dots, \pi|u_d|^2). \tag{3.1}$$

Definition 3.1. For an open set $\Omega \subset \mathbb{R}^d_\zeta$, we call $X_\Omega \coloneqq \mu^{-1}(\Omega) \subset T^*V$ an (open) *toric domain.* We say X_Ω is a *convex toric domain* if $|\Omega| \coloneqq \{\zeta \in \mathbb{R}^d : (|\zeta_1|, \dots, |\zeta_d|) \in \Omega\}$ is convex. We say X_Ω is a *concave toric domain* if $\mathbb{R}^d_{\zeta\geq 0} \setminus \Omega$ is convex.

Remark 3.2. Since the moment map μ has the image $\mathbb{R}^d_{\zeta\geq 0}$, the toric domain X_Ω is determined by $\Omega \cap \mathbb{R}^d_{\zeta\geq 0}$. So we have freedom to choose suitable Ω. For example, we always assume $-\mathbb{R}^d_{\zeta\geq 0} \subset \Omega$. If X_Ω is a convex or a concave toric domain, one can indeed take Ω to be convex or concave (in the usual sense) and satisfying the condition $-\mathbb{R}^d_{\zeta\geq 0} \subset \Omega$. (e.g. replace Ω by $\Omega - \mathbb{R}^d_{\zeta\geq 0}$).

For example, we can take a non-decreasing sequence $a = (a_1, \dots, a_d)$ of positive real numbers, let $\Omega_{D(a)} = \{\zeta \in \mathbb{R}^d_\zeta : \zeta_i < a_i, i \in [d]\}$ and $\Omega_{E(a)} = \{\zeta \in \mathbb{R}^d_\zeta : \frac{\zeta_1}{a_1} + \cdots + \frac{\zeta_d}{a_d} < 1\}$. Then $X_{\Omega_{D(a)}} = D(a)$ is an open poly-disc and $X_{\Omega_{E(a)}} = E(a)$ is an open ellipsoid. Both are convex toric domains.

3.1. Generating function model for microlocal kernel of Toric domains

In [Chi17, Proposition 3.10], Chiu constructs a sheaf quantization of Hamiltonian rotation in all dimensions, particularly for the 2-dimensional φ_z, say $\mathcal{S} \in \mathcal{D}(\mathbb{R}_z \times \mathbb{R}_{q_1} \times \mathbb{R}_{q_2})$. This quantization possesses one more property than we stated for general sheaf quantizations (see (2.3)), namely

$$\mathcal{S} \cong \mathrm{R}\pi_{(q_2,\dots,q_N)!}\mathbb{K}_{\{(z,q_1,\dots,q_{N+1},t):t+\sum_{j=1}^N S_H(z/N,q_j,q_{j+1})\geq 0\}}, \tag{3.2}$$

where we identify q_{N+1} with q_2 after pushforward, N is big enough so that $z/N \in (-1/4, 0) \cup (0, 1/4)$, and S_H is the generating function of the Hamiltonian rotation:

$$S_H(z, q, q') = \frac{q^2 + q'^2}{2\tan(2\pi z)} - \frac{qq'}{\sin(2\pi z)}. \tag{3.3}$$

The formula (3.2) is essential when computing the Chiu-Tamarkin complexes for convex toric domains.

Let

$$\mathcal{T} := \mathcal{S}^{\boxtimes d} = \mathrm{R}s_{t!}^{d}(\mathcal{S}^{\overset{L}{\boxtimes} d}) \in \mathcal{D}(\mathbb{R}_z^d \times V_1 \times V_2), \tag{3.4}$$

where $V_i = \mathbb{R}^d_{\mathbf{q}_i}$. The microsupport estimates show that $\mathcal{T}$ is a sheaf quantization of the standard torus action in the sense of (2.3). As a corollary of Proposition 2.8, we have

Proposition 3.3. *A toric domain X_Ω is dynamically admissible by the distinguished triangle*

$$\widehat{\mathcal{T}} \circ \mathbb{K}_\Omega \to \mathbb{K}_{\Delta_{V^2} \times \{t \geq 0\}} \to \widehat{\mathcal{T}} \circ \mathbb{K}_{\mathbb{R}^d_\zeta \setminus \Omega} \xrightarrow{+1}, \tag{3.5}$$

and the pair of kernels

$$P_{X_\Omega} := \widehat{\mathcal{T}} \circ \mathbb{K}_\Omega, \qquad Q_{X_\Omega} := \widehat{\mathcal{T}} \circ \mathbb{K}_{\mathbb{R}^d_\zeta \setminus \Omega}. \tag{3.6}$$

This pair of microlocal kernels $(P_{X_\Omega}, Q_{X_\Omega})$ constructed from $\mathcal{T}$ is called the *generating function model* of the microlocal kernels associated to toric domains.

Actually, by the microsupport estimate of $\widehat{\mathcal{T}}$ (see (2.4)), if $(\zeta, z, \mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}', t, \tau) \in \dot{SS}(\widehat{\mathcal{T}})$ then we have $\zeta = \mu(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^d_{\zeta \geq 0}$. So, if $\zeta \notin \mathbb{R}^d_{\zeta \geq 0}$ and $(\zeta, z, \mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}', t, \tau) \in SS(\widehat{\mathcal{T}})$, we have $(\mathbf{p}, \mathbf{p}', \tau) = 0$. Accordingly, for any $\zeta \notin \mathbb{R}^d_{\zeta \geq 0}$, we have $SS(\widehat{\mathcal{T}}|_{(\zeta, \mathbf{q}, \mathbf{q}')}) \subset \{\tau = 0\}$ by the microsupport estimate (Theorem 1.4). So $\widehat{\mathcal{T}}|_{(\zeta, \mathbf{q}, \mathbf{q}')} \cong M_\mathbb{R}$ is a constant sheaf over $\mathbb{R}$ by Theorem 1.2 for some $M \in D(\mathbb{K} - \mathrm{Mod})$. As $\widehat{\mathcal{T}}|_{(\zeta, \mathbf{q}, \mathbf{q}')} \in \mathcal{D}(\mathrm{pt})$, and we have $\widehat{\mathcal{T}}|_{(\zeta, \mathbf{q}, \mathbf{q}')} \cong M_\mathbb{R} \cong M_\mathbb{R} \star \mathbb{K}_{[0,\infty)} \cong 0$. We conclude that $\mathrm{supp}(\widehat{\mathcal{T}}) \subset \mathbb{R}^d_{\zeta \geq 0}$.

Consequently, the kernel P_{X_Ω} satisfies

$$P_{X_\Omega} := \widehat{\mathcal{T}} \circ \mathbb{K}_\Omega \cong \mathrm{R}\pi_{\zeta !}(\widehat{\mathcal{T}} \overset{L}{\otimes} \mathbb{K}_{\Omega \times X^2 \times \mathbb{R}_t}) \cong \mathrm{R}\pi_{\zeta !}(\widehat{\mathcal{T}} \overset{L}{\otimes} \mathbb{K}_{(\Omega \cap \mathbb{R}^d_{\zeta \geq 0}) \times X^2 \times \mathbb{R}_t}), \tag{3.7}$$

which only depends on $\Omega \cap \mathbb{R}^d_{\zeta \geq 0}$. So, it is the same as Remark 3.2 that the notation P_{X_Ω} makes sense.

In general, it is complicated to compute the Fourier transform $\widehat{\mathcal{T}}$. However, with the help of associativity of composition and convolution (1.2), we have

$$\widehat{\mathcal{T}} \circ \mathbb{K}_\Omega \cong \mathcal{T} \star \widehat{\mathbb{K}_\Omega}. \tag{3.8}$$

When X_Ω is convex, we can take a suitable Ω, which is convex in the usual sense. Then, the Fourier transform $\widehat{\mathbb{K}_\Omega}$ is easy to compute. Actually, when X_Ω is convex, we have $\widehat{\mathbb{K}_\Omega} \cong \mathbb{K}_{\Omega^\circ}$ by a similar argument with Corollary 2.11, where

$$\Omega^\circ = \{(z,t) : t + \langle z, \zeta\rangle \geq 0, \forall \zeta \in \Omega\}.$$

The assumption $-\mathbb{R}^d_{\zeta\geq 0} \subset \Omega$ shows $\Omega^\circ \subset \mathbb{R}^d_{z\leq 0} \times [0,\infty)$. Then we conclude that when X_Ω is a convex toric domain, we have

$$P_{X_\Omega} \cong \mathcal{T} \star \mathbb{K}_{\Omega^\circ}, \qquad F_\ell(X_\Omega, \mathbb{K}) \cong \mathrm{R}\pi_{z!}\mathrm{R}s^2_{t!}\left(\mathcal{CL}_\ell(\mathcal{T}) \star \mathbb{K}_{\Omega^\circ}\right). \tag{3.9}$$

Example 3.4. Let $a = (a_1, \ldots, a_d)$ be a non-decreasing sequence of positive real numbers.

1) Suppose $\Omega_{D(a)} = \{\zeta : \zeta_i < a_i, i \in [d]\}$, then $X_{\Omega_{D(a)}} = D(a)$ is an open poly-disc. Let P_r be the kernel of the open disc $\{\pi|u|^2 < r\}$ in $\mathbb{C}$, then Corollary 2.11 applies and $P_{D(a)} \cong P_{a_1} \boxtimes \cdots \boxtimes P_{a_d}$.

2) Suppose $\Omega_{E(a)} = \{\zeta : \frac{\zeta_1}{a_1} + \cdots + \frac{\zeta_d}{a_d} < 1\}$, then $X_{\Omega_{E(a)}} = E(a)$ is an open ellipsoid, and $\Omega^\circ_{E(a)} = \{(z,t) : t \geq -a_1 z_1 = \cdots = -a_d z_d \geq 0\}$.

Let $i : \mathbb{R}_z \to \mathbb{R}^d_z$, $z \mapsto (a_1 z, \ldots, a_d z)$, then $\mathbb{K}_{\Omega^\circ_{E(a)}} = \mathrm{R}(i \times \mathrm{Id}_\mathbb{R})_! \mathbb{K}_{\{t \geq -z \geq 0\}}$. Therefore, we have

$$\begin{aligned} P_{E(a)} &\cong \mathcal{T} \star \mathrm{R}(i \times \mathrm{Id}_\mathbb{R})_! \mathbb{K}_{\{t\geq -z\geq 0\}} \\ &\cong ((i \times \mathrm{Id}_\mathbb{R})^{-1}\mathcal{T}) \star \mathbb{K}_{\{t \geq -z \geq 0\}} \cong \widehat{(i \times \mathrm{Id}_\mathbb{R})^{-1}\mathcal{T}} \circ \mathbb{K}_{(-\infty,1)}. \end{aligned}$$

Here we should be careful that, to obtain the second isomorphism, we need to use the explicit formula (3.2) and (3.4).

One can check directly that $(i \times \mathrm{Id}_\mathbb{R})^{-1}\mathcal{T}$ is the sheaf quantization of the diagonal Hamiltonian rotation $\varphi_z(u) = (\exp(\frac{-2i\pi z}{a_1})u_1, \ldots, \exp(\frac{-2i\pi z}{a_d})u_d)$ in the sense of (2.3). In particular, when $a_1 = \cdots = a_d = \pi R^2 > 0$, the construction is the same as Chiu's for balls.

Remark 3.5. For the concave toric domain case, the Fourier transform $\widehat{\mathbb{K}_\Omega}$ is not as simple as the convex case (which is a complex only concentrated in degree 0). Actually, $\widehat{\mathbb{K}_\Omega}$ is represented by a complex of sheaves concentrated in cohomological degree $[0, d]$. Accordingly, the results in the next subsection cannot generalize directly to the concave situation. However, some manual computations of capacities for concave toric domains are still as predicted in Conjecture 0.C.

For toric domains neither convex nor concave, the first example we can consider is an open annulus bounded by two concentric spheres. Then we can take $\Omega = \{x \in \mathbb{R}^d : a < \sum x_i < A\}$. In this case, when $T \geq 0$, we can only extract numerical information about the exterior sphere from $\widehat{\mathbb{K}_\Omega}$. Then we cannot know numerical information for the interior ball. Maybe it is a feature of the present definition of capacities, we expect more understanding to overcome this defect.

3.2. Chiu-Tamarkin complex and capacities of convex toric domains

In this subsection, we focus on convex toric domains, that is, $X_\Omega = \mu^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is an open set such that $\{(\zeta_1, \zeta_d) \in \mathbb{R}^d : (|\zeta_1|, \dots, |\zeta_d|) \in \Omega\}$ is convex. As we discussed in Remark 3.2, we could take a convex Ω such that $\mathbb{R}^d_{\zeta \leq 0} \subset \Omega$. The identity (3.7) shows that such a choice of Ω does not affect the computation of Chiu-Tamarkin complex for X_Ω and we will see this feature again in Remark 3.15.

One can verify that, under such conditions, the polar cone satisfies $\{O\} \times \mathbb{R}_{\geq 0} \subset \Omega^\circ \subset \mathbb{R}^d_{\leq 0} \times \mathbb{R}_{\geq 0}$, where $O \in \mathbb{R}^d$ is the origin. For $T \geq 0$, we set

$$\Omega^\circ_T \coloneqq \Omega^\circ \cap \{t = T\} = \{z \in \mathbb{R}^d : T + \langle z, \zeta \rangle \geq 0, \forall \zeta \in \Omega\}.$$

We also define the function $I(z) = \sum_{i=1}^d \lfloor -z_i \rfloor$, $z \in \mathbb{R}^d$. For a subset $\Sigma \subset \mathbb{R}^d$, we define

$$\|\Sigma\|_\infty = \max_{z \in \Sigma} \|z\|_\infty \quad \text{and } I(\Sigma) = \max_{z \in \Sigma} I(z). \tag{3.10}$$

Then we have $\|\Omega^\circ_T\|_\infty = T\|\Omega^\circ_1\|_\infty$ for $T \geq 0$.

For $x, y \in \mathbb{R}^d$, the segment $\overline{xy}$ is defined as $\{tx + (1-t)y : t \in [0,1]\}$.

Theorem 3.6. *Let $X_\Omega \subset T^*V$ be a convex toric domain and $\ell \in \mathbb{N}_{\geq 2}$. If $0 \leq T < p_\ell / \|\Omega^\circ_1\|_\infty$, we have*

- *For each $Z \in \Omega^\circ_T$, the inclusion of the segment $\overline{OZ} \subset \Omega^\circ_T$ induces a decomposition of the fundamental class $\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}) = u^{I(Z)} \Lambda_{Z,\ell}$ for a non-torsion element $\Lambda_{Z,\ell} \in H^{-2I(Z)} C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$. In particular, $\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ is non-zero.*

• *The minimal cohomology degree of* $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ *is exactly* $-2I(\Omega_T^\circ)$, *i.e.,*

$$H^*C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}) \cong H^{\geq -2I(\Omega_T^\circ)}C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}),$$

and

$$H^{-2I(\Omega_T^\circ)}C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}) \neq 0.$$

• $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ *is a finitely generated* $\mathbb{F}_{p_\ell}[u]$*-module. The free part is isomorphic to* $A = \mathbb{F}_{p_\ell}[u, \theta]$, *so* $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ *is of rank* 2 *over* $\mathbb{F}_{p_\ell}[u]$.

The torsion part is located in cohomology degree $[-2I(\Omega_T^\circ), -1]$. $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ *is torsion free when* X_Ω *is an open ellipsoid.*

Before proving Theorem 3.6, let us use it to compute the capacities $c_k(X_\Omega)$.

Theorem 3.7. *For a convex toric domain* $X_\Omega \subsetneq T^*V$, *we have*

$$c_k(X_\Omega) = \inf \{T \geq 0 : \exists z \in \Omega_T^\circ, I(z) \geq k\}.$$

Proof. Let $S = \{T \geq 0 : \exists z \in \Omega_T^\circ, I(z) \geq k\}$, $L = \inf(S)$.

For $T \in S$, there is $Z \in \Omega_T^\circ$ such that $I(Z) = k$. Consider the closed inclusion of the segment $\overline{OZ} \subset \Omega_T^\circ$. We choose a prime p with $p > T\|\Omega_1^\circ\|_\infty$. Then for all $\ell \in \mathbb{N}_{\geq 2}$ with $p_\ell \geq p$, we have $p_\ell > T\|\Omega_1^\circ\|_\infty$, and Theorem 3.6 shows that the closed inclusion induces a decomposition $\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}) = u^k\Lambda_{Z,\ell}$. So $T \in \mathrm{Spec}(X_\Omega, k)$, and $L \geq c_k(X_\Omega)$.

Conversely, if $T \in \mathrm{Spec}(X_\Omega, k)$, there is a prime p such that for all $\ell \in \mathbb{N}_{\geq 2}$ with $p_\ell \geq p$ there is a $\Lambda_\ell \in H^*C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ such that $\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}) = u^k\Lambda_\ell$. Now, we can take a prime $\ell = p_\ell > p$ big enough such that $T < \ell/\|\Omega_1^\circ\|_\infty$, then $\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ and Λ_ℓ are non-zero. Hence, we have an equation of degree: $0 = |\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})| = 2k + |\Lambda_\ell|$, which shows that $2k = -|\Lambda_\ell|$. Therefore, Theorem 3.6 shows $2k = -|\Lambda_\ell| \leq 2I(\Omega_T^\circ)$. Hence $T \in S$, and $c_k(X_\Omega) \geq L$. □

Here, we test the result by the example of ellipsoids. They are all direct corollaries of Theorem 3.6 and Theorem 3.7.

Corollary 3.8. *Let* $X_\Omega = E = E(a_1, \dots, a_d)$ *be an ellipsoid and* $\ell \in \mathbb{N}_{\geq 2}$. *For* $0 \leq T < p_\ell a_1$, *set* $Z(a) = (-T/a_1, \dots, -T/a_d)$. *We have* $H^*C_T^{\mathbb{Z}/\ell}(E, \mathbb{F}_{p_\ell}) \cong u^{-I(Z(a))}\mathbb{F}_{p_\ell}[u, \theta]$, *the fundamental class is non-zero in all cases, and* $c_k(E) = \min\{T \geq 0 : \sum_{i=1}^d \lfloor T/a_i \rfloor \geq k\}$. *In particular,* $c_k(B_a) = \lceil k/d \rceil a$.

3.3. Cohomology sheaf $\mathcal{CL}_\ell(\mathcal{T})$ for the standard torus action

Recall the results of subsection 3.1, and discussions in subsection 2.3. It is necessary to study the cohomology sheaf $\mathcal{CL}_\ell(\mathcal{T})$ carefully. Recall that $\mathcal{T} = \mathcal{S}^{\circledast d} = \mathrm{R}s^d_{t!}(\mathcal{S}^{\overset{L}{\boxtimes} d})$, where $\mathcal{S}$ is the sheaf quantization of Hamiltonian rotation in dimension 2. Using the Künneth formula and Proposition 2.19, we have

$$\begin{aligned}\mathcal{CL}_\ell(\mathcal{T}) &\cong \mathrm{R}s^\ell_{\underline{z}*}\left(\left((s^\ell_{z_j})^{-1}\mathcal{CL}_\ell(\mathcal{S})\right)^{\circledast d}\right) \cong \mathrm{R}s^\ell_{\underline{z}*}\left((s^\ell_{\underline{z}})^{-1}\left(\mathcal{CL}_\ell(\mathcal{S})\right)^{\circledast d}\right)\\ &\cong \mathrm{R}s^\ell_{\underline{z}*}(s^\ell_{\underline{z}})^{-1}\left(\left(\mathcal{CL}_\ell(\mathcal{S})\right)^{\circledast d}\right) \cong \mathrm{R}s^d_{t!}\left(\mathcal{CL}_\ell(\mathcal{S})\right)^{\circledast d},\end{aligned}$$

where $\underline{z} = (z_1, \ldots, z_d)$. Moreover, an explicit formula for $\mathcal{CL}_\ell(\mathcal{S})$ is obtained by Chiu:

Proposition 3.9. ([Chi17, Formula (38)]) *For all fields $\mathbb{K}$, there exists a (unique) sheaf $\mathcal{E}_\ell \in D_{\mathbb{Z}/\ell}(\mathbb{R}_z)$ such that we have an isomorphism in $D_{\mathbb{Z}/\ell}(\mathbb{R}_z \times \mathbb{R}_t)$*

$$\mathcal{CL}_\ell(\mathcal{S}) \cong \mathcal{E}_\ell \overset{L}{\boxtimes} \mathbb{K}_{[0,\infty)}. \tag{3.11}$$

Moreover, for any $N \in \mathbb{N}$,

$$\mathcal{E}_\ell|_{(-N\ell/4,0)} \cong \mathrm{R}\pi_{\underline{q}!}\mathbb{K}_{\mathcal{W}^N_\ell}, \tag{3.12}$$

with $\underline{q} = (q_1, \ldots, q_{N\ell})$,

$$\mathcal{W}^N_\ell = \{(z, q_1, \ldots, q_{N\ell}) \in (-N\ell/4, 0) \times \mathbb{R}^{N\ell} : \sum_{k\in\mathbb{Z}/N\ell} S_H(z/N\ell, q_k, q_{k+1}) \geq 0\},$$

and

$$S_H(z, q_k, q_{k+1}) = \frac{q_k^2 + q_{k+1}^2}{2\tan(2\pi z)} - \frac{q_k q_{k+1}}{\sin(2\pi z)}.$$

The $\mathbb{Z}/\ell$-action on $\mathcal{E}_\ell$ is induced by the linear action $(q_k) \mapsto (q_{k-N})$ of $\mathbb{Z}/\ell$ on $\mathbb{R}^{N\ell}$, and $\mathbb{Z}/\ell$ acts trivially on $\mathbb{R}_z \times \mathbb{R}_t$.

A disadvantage for the formula (3.12) is that we don't know if the isomorphism can be extended to $z = 0$ since the right hand side is not defined for $z = 0$. Such an extension is necessary for our later computation. So, let us start from an extension of the isomorphism (3.12) to $z = 0$. Notice that $\sin(2\pi z/N\ell) < 0$ for $z/N\ell \in (-1/4, 0)$. One can rewrite $\mathcal{W}_\ell^N$ as follows:

$$\mathcal{W}_\ell^N = \Big\{(z, q_1, \dots, q_{N\ell}) \in (-N\ell/4, 0) \times \mathbb{R}^{N\ell} : \cos(2\pi z/N\ell) \sum_{k\in\mathbb{Z}/N\ell} q_k^2 \leq \sum_{k\in\mathbb{Z}/N\ell} q_k q_{k+1} \Big\}.$$

Let us define

$$Q(z, q_1, \dots, q_{N\ell}) := \sum_{k\in\mathbb{Z}/N\ell\mathbb{Z}} \left(q_k q_{k+1} - \cos(2\pi z/N\ell) q_k^2\right). \tag{3.13}$$

Since $Q(0, q_1, \dots, q_{N\ell})$ is well defined, we can extend the definition of $\mathcal{W}_\ell^N$ (using the same notation) to

$$\mathcal{W}_\ell^N = \{(z, q_1, \dots, q_{N\ell}) \in (-N\ell/4, 0] \times \mathbb{R}^{N\ell} : Q(z, q_1, \dots, q_{N\ell}) \geq 0\}. \tag{3.14}$$

For our convenience, we also set, for $z \in (-N\ell/4, 0]$,

$$\mathcal{W}_\ell^N(z) = \{(q_1, \dots, q_{N\ell}) \in \mathbb{R}^{N\ell} : Q(z, q_1, \dots, q_{N\ell}) \geq 0\}. \tag{3.15}$$

The $\mathbb{Z}/\ell$-action on the extension is the same as the original one.

Now take

$$\mathcal{E}'_\ell := \mathrm{R}\pi_{\underline{q}!} i_! \mathbb{K}_{\mathcal{W}_\ell^N} \in D_{\mathbb{Z}/\ell}((-N\ell/4, +\infty)),$$

where $i : (-N\ell/4, 0] \times \mathbb{R}^{N\ell} \hookrightarrow (-N\ell/4, +\infty) \times \mathbb{R}^{N\ell}$ is the closed inclusion.

By the fundamental inequality, we have $\sum_k q_k^2 \geq \sum_k q_k q_{k+1}$, and it takes equality when $q_1 = \cdots = q_{N\ell}$. So

$$\mathcal{W}_\ell^N(0) = \{(q_1, \dots, q_{N\ell}) \in \mathbb{R}^{N\ell} : q_1 = \cdots = q_{N\ell}\} = \Delta_{\mathbb{R}^{N\ell}}.$$

Then we have that $(\mathcal{E}'_\ell)_0 = \mathrm{R}\Gamma_c(\mathcal{W}_\ell^N(0), \mathbb{K}_{\Delta_{\mathbb{R}^\ell}}) \cong \mathrm{R}\Gamma_c(\Delta_{\mathbb{R}^\ell}, \mathbb{K}_{\Delta_{\mathbb{R}^\ell}})$.

On the other hand, one can check that $\mathcal{CL}_\ell(\mathcal{S})|_{\{z=0\}} = \mathrm{R}\Gamma_c(\Delta_{\mathbb{R}^\ell}, \mathbb{K}_{\Delta_{\mathbb{R}^\ell}}) \overset{L}{\boxtimes} \mathbb{K}_{\{t\geq 0\}}$ by definition of $\mathcal{CL}_\ell(\mathcal{S})$ since $\mathcal{S}|_{\{z=0\}} = \mathbb{K}_{\Delta_{\mathbb{R}^2}} \overset{L}{\boxtimes} \mathbb{K}_{\{t\geq 0\}}$. Therefore, we have $\mathcal{CL}_\ell(\mathcal{S})|_{\{z=0\}} = (\mathcal{E}'_\ell)_0 \overset{L}{\boxtimes} \mathbb{K}_{\{t\geq 0\}}$.

However, stalk-wise isomorphism is not necessary extend to a global one in general. So, we need the following prove to obtain a global extension of the isomorphism (3.12).

Lemma 3.10. *We have an equivariant isomorphism*

$$\mathcal{E}_\ell|_{(-N\ell/4,0]} \cong \mathcal{E}'_\ell|_{(-N\ell/4,0]} = \mathrm{R}\pi_{\underline{q}!}\mathbb{K}_{\mathcal{W}_\ell^N}.$$

Proof. Using Theorem 1.4 and Theorem 1.7, one can show that $SS(\mathcal{E}_\ell), SS((\mathcal{E}_\ell)_{(-N\ell/4,0]}) \subset \{\zeta \leq 0\}$. Now, consider the distinguished triangle

$$\mathrm{R}\Gamma_{[0,\infty)}((\mathcal{E}_\ell)_{(-N\ell/4,0]}) \to (\mathcal{E}_\ell)_{(-N\ell/4,0]} \to \mathrm{R}\Gamma_{(-N\ell/4,0)}((\mathcal{E}_\ell)_{(-N\ell/4,0]}) \xrightarrow{+1} .$$

By definition, we have $\mathrm{supp}(\mathrm{R}\Gamma_{[0,\infty)}((\mathcal{E}_\ell)_{(-N\ell/4,0]})) \subset \{0\}$. On $(-N\ell/4, +\infty)$, the closed set $[0,\infty)$ is defined by the function $f(z) = z$ and $\{f(z) \geq 0\}$. Therefore, by definition of microsupport, $(\mathrm{R}\Gamma_{\{z\geq 0\}}((\mathcal{E}_\ell)_{(-N\ell/4,0]}))_0 \cong 0$ since $df_0 = (0,1) \notin SS((\mathcal{E}_\ell)_{(-N\ell/4,0]})$. So we have $(\mathrm{R}\Gamma_{\{z\geq 0\}}((\mathcal{E}_\ell)_{(-N\ell/4,0]}))_0 \cong 0$ and we have an isomorphism $(\mathcal{E}_\ell)_{(-N\ell/4,0]} \cong \mathrm{R}\Gamma_{(-N\ell/4,0)}((\mathcal{E}_\ell)_{(-N\ell/4,0]})$. This isomorphism holds in the equivariant category since the corresponding morphism is an equivariant morphism.

The argument is purely microlocal, so we also have $(\mathcal{E}'_\ell)_{(-N\ell/4,0]} \cong \mathrm{R}\Gamma_{(-N\ell/4,0)}((\mathcal{E}'_\ell)_{(-N\ell/4,0]})$.

On the other hand, the isomorphism (3.12) and our discussion on $\mathcal{W}_\ell^N$ show that $j^{-1}((\mathcal{E}_\ell)_{(-N\ell/4,0]}) \cong j^{-1}((\mathcal{E}'_\ell)_{(-N\ell/4,0]})$ where j is the open inclusion $(-N\ell/4, 0) \hookrightarrow (-N\ell/4, \infty)$. Therefore, the natural isomorphism $\mathrm{R}j_*j^{-1} \cong \mathrm{R}\Gamma_{(-N\ell/4,0)}$ shows that

$$(\mathcal{E}_\ell)_{(-N\ell/4,0]} \cong \mathrm{R}j_*j^{-1}((\mathcal{E}_\ell)_{(-N\ell/4,0]}) \cong \mathrm{R}j_*j^{-1}\left((\mathcal{E}'_\ell)_{(-N\ell/4,0]}\right) \cong \left(\mathcal{E}'_\ell\right)_{(-N\ell/4,0]} .$$

Finally, we conclude by restricting the isomorphism to $(-N\ell/4, 0]$ and the definition of $\mathcal{E}'_\ell$. □

Topology of $\mathcal{W}_\ell^N(z)$: We know that $(\mathcal{E}_\ell)_z \cong \mathrm{R}\Gamma_c(\mathcal{W}_\ell^N(z), \mathbb{K})$ if $-N\ell/4 < z \leq 0$ (Lemma 3.10). For a fixed $z \in (-N\ell/4, 0]$, the function $Q_z(q_1, \dots, q_{N\ell}) = Q(z, q_1, \dots, q_{N\ell})$ is a quadratic form by (3.13). Therefore, it is easy to study the topology of $\mathcal{W}_\ell^N(z) = \{(q_1, \dots, q_{N\ell}) \in \mathbb{R}^{N\ell} : Q_z \geq 0\}$. The matrix of Q_z in the standard basis is a circulant matrix

$$A_z = \begin{pmatrix} -\cos(\frac{2\pi z}{N\ell}) & \frac{1}{2} & 0 & \cdots & \frac{1}{2} \\ \frac{1}{2} & -\cos(\frac{2\pi z}{N\ell}) & \frac{1}{2} & \cdots & 0 \\ 0 & \frac{1}{2} & -\cos(\frac{2\pi z}{N\ell}) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{2} & 0 & 0 & \cdots & -\cos(\frac{2\pi z}{N\ell}) \end{pmatrix}.$$

So one can diagonalize A_z unitarily using the discrete Fourier transform

$$(\omega^{(i-1)(j-1)})_{i,j=\mathbb{Z}/N\ell},$$

where ω is a primitive $N\ell^{th}$ root of unity. Therefore, the eigenvalues of A_z are

$$\begin{aligned}\lambda_k(z) &= \operatorname{Re}\left(\exp\left(\frac{2\pi k\sqrt{-1}}{N\ell}\right)\right) - \cos\left(\frac{2\pi z}{N\ell}\right) \\ &= \cos\left(\frac{2\pi k}{N\ell}\right) - \cos\left(\frac{2\pi z}{N\ell}\right),\end{aligned} \tag{3.16}$$

where $k \in \mathbb{Z}/N\ell$.

We always have $\lambda_0(z) = 1 - \cos\left(\frac{2\pi z}{N\ell}\right) \geq 0$. It is direct to see that $\lambda_k(z) = \lambda_{N\ell-k}(z)$ for $k = 1, \ldots, N\ell - 1$. So, for $k \geq 1$, we need to consider two situations:

(a) If $N\ell$ is odd. For $k = 1, \ldots, (N\ell-1)/2$, $\lambda_k(z) \geq 0$ if $k \leq \lfloor -z \rfloor$. Therefore, in this case, A_z admits $\#\{k \in \mathbb{Z}/N\ell : \lambda_k \geq 0\} = 1 + 2\lfloor -z \rfloor$ non-negative eigenvalues.

(b) If $N\ell$ is even. The eigenvalue $\lambda_{N\ell/2}(z) = -1 - \cos\left(\frac{2\pi z}{N\ell}\right) < 0$ since $z > -N\ell/4$. For $k = 1, \ldots, (N\ell/2) - 1$, $\lambda_k(z) \geq 0$ if $k \leq \lfloor -z \rfloor$. Therefore, in this case, A_z also admits $\#\{k \in \mathbb{Z}/N\ell : \lambda_k \geq 0\} = 1 + 2\lfloor -z \rfloor$ non-negative eigenvalues.

In any case, we have that A_z admits $\#\{k \in \mathbb{Z}/N\ell : \lambda_k \geq 0\} = 1 + 2\lfloor -z \rfloor$ non-negative eigenvalues.

Therefore, $\mathcal{W}_\ell^N(z) = \{Q_z \geq 0\}$ is a quadratic cone of index $1 + 2\lfloor -z \rfloor$. In particular, $\mathcal{W}_\ell^N(z) = \{Q_z \geq 0\}$ is properly homotopic to a vector space $\mathbb{R}^{1+2\lfloor -z \rfloor}$.

Now we can describe the non-equivariant structure of $\mathcal{E}_\ell|_{(-\infty,0]}$. Here, we forget its equivariant structure and use the same notation $\mathcal{E}_\ell|_{(-\infty,0]}$. In particular, $\mathcal{E}_\ell|_{(-\infty,0]} \cong \mathcal{E}_1|_{(-\infty,0]}$ non-equivariantly. Consider $\pi_{\underline{q}} : \mathcal{W}_\ell^N \to (-N\ell/4, 0]$ for $N\ell$ big enough, it restricts to a proper homotopical fiber bundle with fiber $\mathbb{R}^{1+2n}$ over each interval $(-n-1, -n]$ for $n \in \mathbb{N}_{\geq 0}$, and $n + 1 < N\ell/4$. Therefore, we conclude that $\mathcal{E}_\ell|_{(-n-1,-n]} \cong \mathbb{K}_{(-n-1,-n]}[-1-2n]$. On the other hand, in the non-equivariant derived category, $\mathbb{K}_{(x,y]}$ and $\mathbb{K}_{(z,w]}[2]$ has no non-trivial extension if $\mathbb{K}$ is a field. Therefore, $(\mathcal{E}_\ell)_{(-n-1,-n]}$ has no non-trivial extension for different n. In conclusion, we have

Proposition 3.11. *For all fields $\mathbb{K}$ and for all $\ell \in \mathbb{N}$, we have the decomposition in the non-equivariant derived category $D((-\infty, 0])$:*

$$\mathcal{E}_\ell|_{(-\infty,0]} \cong \bigoplus_{n\in\mathbb{N}_{\geq 0}} \mathbb{K}_{(-n-1,-n]}[-1-2n].$$

To describe the $\mathbb{Z}/\ell$-action on $\mathcal{W}_\ell^N(z)$, it is better to consider the diagonal form of Q_z.

Let $x_k = (q_1, \dots, q_{N\ell})(1, \omega^k, \omega^{2k}, \dots, \omega^{(N\ell-1)k})^t \in \mathbb{C}$, $k \in \mathbb{Z}/N\ell$. They are coordinates after diagonalization using the discrete Fourier transform. As ω is a root of unity, we have that $x_k = \overline{x_{N\ell-k}}$. In particular, x_0 is a real number. Also recall that $\lambda_k(z) = \lambda_{N\ell-k}(z)$. Then the diagonal form of Q_z is

$$\begin{aligned} &Q_z(x_0, x_1, \dots, x_{N\ell-1}) = \lambda_0(z)x_0^2 + \sum_{k=1}^{N\ell-1} \lambda_k(z)|x_k|^2, \\ &(x_0, x_1, \dots, x_{N\ell-1}) \in \mathbb{R} \times \mathbb{C}^{N\ell-1}. \end{aligned} \tag{3.17}$$

Notice that the discrete Fourier transform that we applied is a complex linear transform, it is easier to work in complex coordinates. However, the constrains $x_k = \overline{x_{N\ell-k}}$ shows that actually we only have half independent complex coordinates, so the real dimension here is still $N\ell$. To our convenience in formulating the action, we still use the complex coordinates. We also need to discuss parity of $N\ell$. Since N is chosen arbitrarily, we can always assume N is odd. Then the parity of $N\ell$ is the parity of ℓ.

If ℓ is odd, then the diagonal form is

$$\begin{aligned} &Q_z(x_0, x_1, \dots, x_{(N\ell-1)/2}) = \lambda_0(z)x_0^2 + 2\sum_{k=1}^{(N\ell-1)/2} \lambda_k(z)|x_k|^2, \\ &(x_0, x_1, \dots, x_{(N\ell-1)/2}) \in \mathbb{R} \times \mathbb{C}^{(N\ell-1)/2} \cong \mathbb{R}^{N\ell}. \end{aligned} \tag{3.18}$$

If ℓ is even, then the diagonal form is

$$\begin{aligned} &Q_z(x_0, x_1, \dots, x_{N\ell/2-1}, x_{N\ell/2}) \\ =&\lambda_0(z)x_0^2 + 2\sum_{k=1}^{N\ell/2-1} \lambda_k(z)|x_k|^2 + \lambda_{N\ell/2}(z)|x_{N\ell/2}|^2, \\ &(x_0, x_1, \dots, x_{N\ell/2-1}, x_{N\ell/2}) \in \mathbb{R} \times \mathbb{C}^{N\ell/2-1} \times \mathbb{R} \cong \mathbb{R}^{N\ell}. \end{aligned} \tag{3.19}$$

Now, the action is easier to describe under the diagonal form. By definition of x_k, we have $x_k = \sum_{i\in\mathbb{Z}/N\ell} q_{i+1}\omega^{ik}$. The $\mathbb{Z}/\ell$-action is given by $(q_i) \mapsto (q_{i-N})$.

Then we have

$$x_k = \sum_{i\in\mathbb{Z}/N\ell} q_{i+1}\omega^{ik} \mapsto \sum_{i\in\mathbb{Z}/N\ell} q_{i+1-N}\omega^{ik}$$
$$= \omega^{kN} \sum_{i\in\mathbb{Z}/N\ell} q_{i+1-N}\omega^{(i-N)k} = \omega^{kN} x_k.$$

Therefore, the $\mathbb{Z}/\ell$-action on the diagonal form is as follows: if we take $\mu = \omega^N$ a primitive ℓ^{th} root of unity, then

$$\mu \cdot (x_k)_k = (\mu^k x_k)_k, \tag{3.20}$$

where $k = 0, 1, \ldots, N\ell/2 - 1$ if ℓ is odd and $k = 0, 1, \ldots, N\ell/2$ if ℓ is even.

Consequently, the fixed point sets $\big(\mathcal{W}_\ell^N(z)\big)^{\mathbb{Z}/\ell}$ is again a quadratic cone, whose index is $1 + 2\lfloor -z/\ell \rfloor$. The diagonal $\Delta_{\mathbb{R}^{N\ell}}$ is given by $\{(x_0, 0, \ldots, 0) : x_0 \in \mathbb{R}\}$ in diagonal form, it is a subset of $\big(\mathcal{W}_\ell^N(z)\big)^{\mathbb{Z}/\ell}$.

Finally, we return to the isomorphism of Proposition 3.11. Take $z' \leq z \leq 0$. Since $SS(\mathcal{E}_\ell) \subset \{\zeta \leq 0\}$, the microlocal Morse lemma (Corollary 1.6) shows that $\mathrm{R}\Gamma(\mathbb{R}, (\mathcal{E}_\ell)_{[z,0]}) \cong (\mathcal{E}_\ell)_z$. Then there is a natural morphism

$$(\mathcal{E}_\ell)_{z'} \cong \mathrm{R}\Gamma(\mathbb{R}, (\mathcal{E}_\ell)_{[z',0]}) \to \mathrm{R}\Gamma(\mathbb{R}, (\mathcal{E}_\ell)_{[z,0]}) \cong (\mathcal{E}_\ell)_z.$$

On the other hand, the isomorphism in Lemma 3.10 shows that $(\mathcal{E}_\ell)_z \cong \mathrm{R}\Gamma_c(\mathcal{W}_\ell^N(z), \mathbb{K}) \cong \mathrm{R}\Gamma_c(\mathbb{R}^{1+2\lfloor -z\rfloor}, \mathbb{K})$. Then the natural morphism above is given by

$$\mathrm{R}\Gamma_c(\mathbb{R}^{1+2\lfloor -z'\rfloor}, \mathbb{K}) \to \mathrm{R}\Gamma_c(\mathbb{R}^{1+2\lfloor -z\rfloor}, \mathbb{K}).$$

The decomposition, Proposition 3.11, tells us that the natural morphism is 0 in the non-equivariant category.

In the equivariant category, the morphism is induced from a vector bundle

$$\mathbb{R}^{1+2\lfloor -z'\rfloor} \times_{\mathbb{Z}/\ell} S^\infty \to \mathbb{R}^{1+2\lfloor -z\rfloor} \times_{\mathbb{Z}/\ell} S^\infty,$$

which is a lifting of the following vector bundle

$$\mathbb{R}^{1+2\lfloor -z'\rfloor} \times_{S^1} S^\infty \to \mathbb{R}^{1+2\lfloor -z\rfloor} \times_{S^1} S^\infty$$

via the natural restriction $\mathbb{Z}/\ell \subset S^1$.

So, in the S^1-equivariant derived category, the morphism is given by the mod $\mathbb{K}$ reduction of the ($\mathbb{Z}$-coefficient) top Chern class for the second vector bundle, which is $(\lfloor -z'\rfloor!/\lfloor -z\rfloor!)u^{\lfloor -z'\rfloor - \lfloor -z\rfloor} \in \mathrm{Ext}^*_{S^1}(\mathbb{K}, \mathbb{K})$, which is non-zero. After restricting to the $\mathbb{Z}/\ell$-equivariant derived category, the morphism is non-zero for a suitable reduction in a finite field $\mathbb{K}$. For example, we could require $0 < \lfloor -z' \rfloor < \mathrm{char}(\mathbb{K})$ to make sure that the morphism is non-zero.

The higher dimension ($d \geq 2$) case: Now, we start to discuss the higher dimension situation. We already know that $\mathcal{CL}_\ell(\mathcal{T}) \cong \mathcal{CL}_\ell(\mathcal{S})^{\boxtimes d}$. Then Proposition 3.9 shows that

$$\mathcal{CL}_\ell(\mathcal{T}) \cong \mathcal{E}_\ell^{\overset{L}{\boxtimes} d} \overset{L}{\boxtimes} \mathbb{K}_{\{t \geq 0\}}. \tag{3.21}$$

As the decomposition indicated in Proposition 3.11, $\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}|_{\{z \leq 0\}}$ has a decomposition on $\{z \leq 0\}$ indexed by lattice points. Besides, we also have a topological description of $\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}|_{\{z \leq 0\}}$. Let us first discuss the topological description and then state the decomposition. Since we have d copies of $\mathcal{E}_\ell$, it is convenient to denote $\mathbf{q} = (q^1, \ldots, q^d) \in \mathbb{R}^d =: V_{\mathbf{q}}$. Then Lemma 3.10 shows us

$$\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}\Big|_{(-N\ell/4,0]^d} \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{\prod_{i=1} \mathcal{W}_{\ell,i}^N}, \tag{3.22}$$

where $\mathcal{W}_{\ell,i}^N$ means the ith copy of one $\mathcal{W}_\ell^N$, $i \in [d] = \{1, \ldots, d\}$, $\underline{\mathbf{q}} = (\mathbf{q}_1, \ldots, \mathbf{q}_{N\ell})$ and $\mathbf{q}_k = (q_k^1, \ldots, q_k^d)$. Let $z = (z_1, \ldots, z_d)$, we also define

$$\begin{aligned}
{}^d\mathcal{W}_\ell^N := \prod_{i=1}^d \mathcal{W}_{\ell,i}^N = \{(z, \mathbf{q}_1, \ldots, \mathbf{q}_{N\ell}) \in (-N\ell/4, 0]^d \times V^{N\ell} :& \\
Q_{z_i}((q_k^i)_{k \in [N\ell]}) \geq 0,\ i \in [d]\},& \\
{}^d\mathcal{W}_\ell^N(z) := \prod_{i=1}^d \mathcal{W}_{\ell,i}^N(z_i) = \{(\mathbf{q}_1, \ldots, \mathbf{q}_{N\ell}) \in V^{N\ell} :& \\
Q_{z_i}((q_k^i)_{k \in [N\ell]}) \geq 0,\ i \in [d]\}.&
\end{aligned}$$

The group $\mathbb{Z}/\ell$ acts on each $\mathcal{W}_\ell^N$ via $(q_k^i)_{k \in [N\ell]} \mapsto (q_{k-N}^i)_{k \in [N\ell]}$. Therefore, $\mathbb{Z}/\ell$ acts diagonally on ${}^d\mathcal{W}_\ell^N$ via $(\mathbf{q}_k)_{k \in [N\ell]} \mapsto (\mathbf{q}_{k-N})_{k \in [N\ell]}$.

The diagonalization applies for each $i \in [d]$, and then on ${}^d\mathcal{W}_\ell^N(z)$. We set $\mathbf{x}_k = (x_k^1, \ldots, x_k^d)$ and $\mathbf{x}^i = (x_1^i, \ldots, x_{N\ell}^i)$, then the coordinates of ${}^d\mathcal{W}_\ell^N(z)$ after diagonalization are $(x_k^i)_{i,k} = (\mathbf{x}_k)_k = (\mathbf{x}^i)_i$, where $k = 0, 1, \ldots, (N\ell - 1)/2$ if ℓ is odd and $k = 0, 1, \ldots, N\ell/2$ if ℓ is even.

So for each $z = (z_1, \ldots, z_d) \in (-N\ell/4, 0]^d$, the space ${}^d\mathcal{W}_\ell^N(z)$ is a product of quadratic cones of indices $1 + 2\lfloor -z_i \rfloor$ respectively, and then ${}^d\mathcal{W}_\ell^N(z)$

is properly homotopic to a quadratic cone of index $d + 2I(z)$, where $I(z) = \sum_{i=1}^{d} \lfloor -z_i \rfloor$. Therefore, ${}^d\mathcal{W}_\ell^N(z)$ is properly homotopic to $\mathbb{R}^{d+2I(z)}$ and a refinement of this fact will be proven in Lemma 3.17.

The fixed point sets $\left({}^d\mathcal{W}_\ell^N(z)\right)^{\mathbb{Z}/\ell}$ is also properly homotopic to a quadratic cone of index $d + 2I(z/\ell)$. The diagonal $\Delta_{V^{N\ell}}$ is given by $\{(x_k^i)_{i,k} : \forall i, \forall k \neq 0, x_k^i = 0, x_0^i \in \mathbb{R}\}$ in diagonal form, it is a subset of $\left({}^d\mathcal{W}_\ell^N(z)\right)^{\mathbb{Z}/\ell}$.

To be clear, let us set some higher dimensional interval notation. For $x, y \in \mathbb{R}^d$, we let $(x, y] = \prod_{i=1}^{d}(x_i, y_i]$ be the half-open cube from x to y. We can define half-open cubes $[x, y)$, open cubes (x, y), and closed cubes $[x, y]$ in the same way. Recall that we use $\overline{xy}$ to denote the segment between x, y; only when $d = 1$, we have $\overline{xy} = [x, y]$. Also, recall $O \in \mathbb{R}^d$ is the origin, and we set $\mathbb{1} = (1, \dots, 1)$ and $e_i = (\delta_{ij})_{j=1}^d$ where δ_{ij} stands for the Kronecker symbol.

Then either our topology description of ${}^d\mathcal{W}_\ell^N$ or the decomposition result Proposition 3.11 shows that

Lemma 3.12. *For each $z \leq 0$, we have the equivariant isomorphism*

$$(\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_z \cong \mathrm{R}\Gamma_c(\mathbb{R}^{d+2I(z)}, \mathbb{K}) \cong \mathbb{K}[-d - 2I(z)].$$

In the non-equivariant derived category, we have a decomposition as follows:

$$\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}|_{\{z \leq 0\}} \cong \bigoplus_{v \in \mathbb{N}_0^d} \mathbb{K}_{(-v-\mathbb{1}, -v]}[-d - 2I(-v)].$$

In the equivariant derived category, for $z', z \in (-\infty, 0]^d$, if $z_i' \leq z_i$ for all $i \in [d]$, the natural morphism,

$$\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}|_{z'} \cong \mathbb{K}[-d - 2I(z')] \to \mathcal{E}_\ell^{\overset{L}{\boxtimes} d}|_z \cong \mathbb{K}[-d - 2I(z)],$$

is induced by the mod-$\mathbb{K}$ reduction of the top Chern class of the vector bundle

$$\mathbb{R}^{d+2I(z')} \times_{S^1} ES^\infty \to \mathbb{R}^{d+2I(z)} \times_{S^1} S^\infty,$$

where S^1 acts on $\mathbb{R}^d$ trivially, and acts on $\mathbb{R}^{2I(z)}$ by the weight $((1, \dots, \lfloor -z_i \rfloor))_{i \in [d]}$.

Propagation and γ-topology Finally, let us describe a propagation phenomena of $\mathcal{E}_\ell$. It is simple but crucial for our later application. Notice that, for a given $z \in (-N\ell/4, 0]$, the map $z \mapsto \mathcal{W}_\ell^N(z)$ is a decreasing map with respect to the inclusion order. Microlocally, it means that

$SS(\mathcal{E}_\ell) \subset \{\zeta \leq 0\}$, which is already known to us as a general fact from the microsupport estimate (by Corollary 1.6 for example). We have, for $z \leq 0$, that

$$(\mathcal{E}_\ell)_z \cong \mathrm{R}\Gamma_c(\mathbb{R}, (\mathcal{E}_\ell)_{[z,0]}) \cong \mathbb{K}[-1 - 2\lfloor -z \rfloor].$$

In higher dimension, the same thing still happens. For $z \in (-\infty, 0]^d$, we can compute directly, using Lemma 3.10, to see that

$$(\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_z \cong \mathrm{R}\Gamma_c(\mathbb{R}^d_z, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{[z,O]}) \cong \mathbb{K}[-d - 2I(z)]. \tag{3.23}$$

However, as $SS(\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}) \subset \mathbb{R}^d_z \times (-\infty, 0]^d_\zeta$ and $[z, O] = (\{z\} + [0, \infty)^d) \cap (-\infty, 0]^d$, the isomorphisms (3.23) can also be obtained pure microlocally.

For a closed proper convex cone $\gamma \subset \mathbb{R}^d$, we can consider the γ-topology on $\mathbb{R}^d$. We refer to [KS90, Section 3.5, Section 5.2] and [KS18] for more about the definition and sheaf theory related to γ-topology. A closed subset $Z \subset \mathbb{R}^d$ is γ-closed if $Z = Z - \gamma$. Now, consider the induced topology of the γ-topology on γ. Then the notation $\Sigma_\gamma = (\Sigma - \gamma) \cap \gamma$ is exactly the closure of the γ-topology for a closed set $\Sigma \subset \gamma$. So, for a closed subset $\Sigma \subset \gamma$, we say Σ_γ the γ-*closure* of Σ and we say Σ is γ-*closed* if $\Sigma_\gamma = \Sigma$.

Now, let us take a sheaf $F \in D(\mathbb{R}^d)$ satisfying $SS(F) \subset \mathbb{R}^d \times (-\gamma)$. Then we claim that if Σ is compact and convex, we have that

$$\mathrm{R}\Gamma_c(\mathbb{R}^d_z, F_\Sigma) \cong \mathrm{R}\Gamma_c(\mathbb{R}^d_z, F_{\Sigma_\gamma}). \tag{3.24}$$

We can give a *proof* of (3.24) as follows: The microsupport $SS(F) \subset \mathbb{R}^d \times (-\gamma)$ together with the microlocal cut-off lemma [KS90, Proposition 5.2.3] shows that F_γ is a $-\gamma$-sheaf on $\mathbb{R}^d_z$, i.e. F_γ is pullbacked from a sheaf on $\mathbb{R}^d_z$ equipped with the $-\gamma$-topology. Then its global section over Σ is isomorphic to the global section over the γ-closure Σ_γ by [KS90, Proposition 3.5].

Now, as $SS(\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}) \subset \mathbb{R}^d_z \times (-\infty, 0]^d_\zeta$, we can take $\gamma = (-\infty, 0]^d$, which is a proper convex cone. So, we can talk about Σ_γ for a closed subset $\Sigma \subset \gamma$. For example, $\{z\}_\gamma = [z, O]$ for $z \in \gamma$. Then we apply (3.24) to $\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}$ to obtain the first isomorphism of (3.23) and the stronger result: for a compact and convex set Σ, we have

$$\mathrm{R}\Gamma_c(\mathbb{R}^d_z, (\mathcal{E}_\ell)_\Sigma). \cong \mathrm{R}\Gamma_c(\mathbb{R}^d_z, (\mathcal{E}_\ell)_{\Sigma_\gamma}). \tag{3.25}$$

3.4. Proof of Theorem 3.6

In this subsection, we will prove the structure theorem.

Idea and sketch of the proof: We present $(F_\ell(X_\Omega, \mathbb{K}))_T$ as $\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\Omega_T^\circ}\right)$, to which we can apply the results in subsection 3.3. Now, consider the inclusion sequence $\{O\} \subset \overline{ZO} \subset \Omega_T^\circ$, then we have a commutative diagram.

$$
\begin{array}{ccccc}
\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\Omega_T^\circ}\right) & \longrightarrow & \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\overline{ZO}}\right) & \xrightarrow{\cong} & \mathbb{K}[-d-2I(Z)] \\
& \searrow & \downarrow & & \downarrow {\scriptstyle k_Z u^{I(Z)}} \\
& & \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{O}\right) & \xrightarrow{\cong} & \mathbb{K}[-d]
\end{array}
$$

By definition, the inclined arrow composed with the bottom isomorphism gives the fundamental class, and we call the upper horizontal arrow (up to constant) $\Lambda_{Z,\ell}$. Lemma 3.12 shows that (up to a constant k_Z) the vertical morphism is $u^{I(Z)}$. Eventually, we absorb the constant into $\Lambda_{Z,\ell}$ since the constant is uniquely determined by Z and ℓ. The commutative diagram induces a decomposition $\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{K}) = u^{I(Z)}\Lambda_{Z,\ell}$. In particular, the presence of $\Lambda_{Z,\ell}$ shows us the minimal cohomology degree is smaller than $-2I(Z)$ for all $Z \in \Omega_T^\circ$.

To achieve the non-torsioness, we need to prove that the fundamental class $\eta_T^{\mathbb{Z}/\ell}(U, \mathbb{K})$, a degree 0 morphism, is non-zero. We have two approaches. The easiest one is to take a small ball $B \subset U$, and then we apply the computation for balls (which can be derived directly from Lemma 3.12). The harder one is that we study its cocone, which is computed by homology of a union of finite dimensional manifolds.

I will discuss the harder approach since it provides us with more structural results, for example, rank and degree distribution of torsion elements. We will argue by a localization trick. In particular, we show that $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ is a finitely generated module over $\mathbb{F}_{p_\ell}[u]$ whose free part is of rank 2. Then the argument also shows that torsion cannot happen in non-negative degrees.

Finally, we study further the cocone of the fundamental class to show that the minimal cohomology degree is greater than $-2I(\Omega_T^\circ)$.

Therefore, our technical discussion will focus on the formula for $\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_W\right)$ for a locally closed set $W \subset \Omega_T^\circ$, and its minimal degree estimate. We will organize our arguments in the following way:

• We first compute $(F_\ell(X_\Omega, \mathbb{K}))_T$ using its isomorphism with $\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\Omega_T^\circ}\right)$, where $\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}$ is discussed in the last subsection. Consequently, we derive a similar formula for the cocone of the fundamental class, i.e. $\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\Omega_T^\circ \setminus O}\right)$. Then the result of the last subsection will reduce them to a cohomology of a topological space $\mathcal{W}_\ell^N(\Omega_T^\circ)$ (see (3.29) later for its definition). We will achieve the targets in Lemma 3.14.

• Recall the lattice decomposition (Proposition 3.11) of the sheaf $\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}$. We hope to utilize the lattice description to obtain a minimal degree estimate for the cocone of the fundamental class. A problem here is that we are computing cohomology of sheaves over Ω_T°, while Ω_T° is usually curved. So, our idea is to decompose Ω_T° into "almost cubes", which are introduced in Lemma 3.16. Next, we will study the proper homotopy type of $\mathcal{W}_\ell^N(\Omega_T^\circ)$ in the case that Ω_T° is an almost cube. This is Lemma 3.17.

• Finally, we use the computation for almost cubes as an induction step to obtain the minimal degree estimate in general. This is done using the Mayer–Vietoris sequence in Lemma 3.18. After that, we will finish the proof of Theorem 3.6.

Remark 3.13. A technical fact is that in the induction process of the minimal degree estimate, we have to deal with some sets that are not necessarily convex. However they are γ-closed. So, we will present the result for γ-closed set Σ, not only Ω_T°, from the beginning in the following.

Preliminary lemmas: For a convex toric domain X_Ω, by the discussion following (3.8), we have $\Omega^\circ \subset \gamma^d \times [0, \infty)$. Then (2.17) and (3.21) show that

$$
\begin{aligned}
F_\ell(X_\Omega, \mathbb{K}) &\cong \mathrm{R}\pi_{z!}\mathrm{R}s_{t!}^2\left(\mathcal{E}_\ell^{\overset{L}{\boxtimes} d} \overset{L}{\boxtimes} \mathbb{K}_{\{t_1 \geq 0\}} \overset{L}{\otimes} \pi_{t_1}^{-1}\mathbb{K}_{\Omega^\circ}\right) \\
&\cong \mathrm{R}\pi_{z!}\left[(\mathcal{E}_\ell^{\overset{L}{\boxtimes} d} \overset{L}{\boxtimes} \mathbb{K}_{\{t \geq 0\}})_{\Omega^\circ}\right].
\end{aligned} \tag{3.26}
$$

Therefore, we conclude that

$$
(F_\ell(X_\Omega, \mathbb{K}))_T \cong \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\Omega_T^\circ}\right). \tag{3.27}
$$

In particular, for $X_{\mathbb{R}^d} = T^*V$, we have

$$(F_\ell(T^*V, \mathbb{K}))_T \cong \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_O\right) \cong \mathbb{K}[-d].$$

Then, by definition, the fundamental class is

$$\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\Omega_T^\circ}\right) \to \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_O\right) \cong \mathbb{K}[-d].$$

For $Z \in \Omega_T^\circ$, we apply (3.25) for the segment $\Sigma = \overline{ZO}$, then we have (recall that $[Z, O]$ denotes a cube here)

$$\mathrm{R}\Gamma_c(\mathbb{R}_z^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes}\ell})_{\overline{ZO}}) \cong \mathrm{R}\Gamma_c(\mathbb{R}_z^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes}\ell})_{[Z,O]}) \cong \mathbb{K}[-d - 2I(Z)],$$

since $[Z, O] = \overline{ZO}_\gamma$. Now, we can embed the fundamental class into an excision triangle:

$$\begin{aligned}&\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\Omega_T^\circ \setminus O}\right) \to \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\Omega_T^\circ}\right)\\ &\xrightarrow{\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{K})} \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_O\right) \xrightarrow{+1} .\end{aligned}$$

Both Ω_T° and $\overline{ZO}$ are compact convex. We would like to apply the isomorphism (3.22) to compute the cohomology of $\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}$ in term of ${}^d\mathcal{W}_\ell^N$.

Assumption: For any compact subset $\Sigma \subset \gamma$, we will fix an odd integer $N = N(\Sigma) > 0$ and a positive number $\varepsilon > 0$ such that $\Sigma \subset [-N\ell/4 - \varepsilon, 0]^d$. The existence of N and ε is ensured by the compactness of Σ.

Lemma 3.14. *For a compact set $\Sigma \subset \gamma$ such that $\Sigma \cap [x, y]$ is empty or contractible for all $x \le y$, $x, y \in \gamma$ (recall here, $[x, y]$ means the closed cube from x to y). We have*

$$\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_\Sigma\right) \cong \mathrm{R}\Gamma_c\left(\mathcal{W}_\ell^N(\Sigma), \mathbb{K}\right), \tag{3.28}$$

where

$$\mathcal{W}_\ell^N(\Sigma) = \bigcup_{z \in \Sigma} {}^d\mathcal{W}_\ell^N(z) = \pi_z\left({}^d\mathcal{W}_\ell^N \cap (\Sigma \times V^{N\ell})\right). \tag{3.29}$$

As $\Sigma = \Omega_T^\circ$ is convex for $T \geq 0$, we have, in particular

$$(3.30)\qquad \begin{aligned} (F_\ell(X_\Omega, \mathbb{K}))_T &\cong \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\Omega_T^\circ}\right) \cong \mathrm{R}\Gamma_c\left(\mathcal{W}_\ell^N(\Omega_T^\circ), \mathbb{K}\right), \\ \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\Omega_T^\circ \setminus O)}\right) &\cong \mathrm{R}\Gamma_c\left(\mathcal{W}_\ell^N(\Omega_T^\circ) \setminus \Delta_{V^{N\ell}}, \mathbb{K}\right). \end{aligned}$$

Proof. For $N = N(\Sigma) > 0$ and $\varepsilon > 0$ such that $\Sigma \subset [-N\ell/4 - \varepsilon, 0]^d$, we have the isomorphism (3.22) $\mathcal{E}_\ell^{\overset{L}{\boxtimes} d}|_{[-N\ell/4-\varepsilon,0]^d} \cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{{}^d\mathcal{W}_\ell^N \cap ([-N\ell/4-\varepsilon,0]^d \times V^{N\ell})}$, and then we obtain

$$\begin{aligned} \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_\Sigma\right) &\cong \mathrm{R}\pi_{z!}\left((\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{{}^d\mathcal{W}_\ell^N})_\Sigma\right) \\ &\cong \mathrm{R}\pi_{z!}\mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathbb{K}_{{}^d\mathcal{W}_\ell^N \cap (\Sigma \times V^{N\ell})} \\ &\cong \mathrm{R}\pi_{\underline{\mathbf{q}}!}\mathrm{R}\pi_{z!}\mathbb{K}_{{}^d\mathcal{W}_\ell^N \cap (\Sigma \times V^{N\ell})}. \end{aligned}$$

Claim: When restricted to ${}^d\mathcal{W}_\ell^N \cap (\Sigma \times V^{N\ell})$, the fiber of π_z is compact and contractible if it is non-empty. Indeed, Chiu proved, in the Lemma 4.10 of [Chi17], that the fibers of the restriction of π_{z_i} on $\mathcal{W}_\ell^N \cap ([-N\ell/4 - \varepsilon, 0] \times \mathbb{R}^{N\ell})$ are closed intervals. So the fibers of the restriction of π_z on ${}^d\mathcal{W}_\ell^N \cap ([-N\ell/4 - \varepsilon, 0] \times \mathbb{R}^{N\ell})$ are closed cubes. Hence, the fibers of the restriction of π_z on ${}^d\mathcal{W}_\ell^N \cap (\Sigma \times V^{N\ell})$ are intersections of closed cubes and Σ, which are either empty or compact and contractible by assumption.

Consequently, the Vietoris-Begel theorem implies

$$\mathrm{R}\pi_{z!}\mathbb{K}_{{}^d\mathcal{W}_\ell^N \cap (\Sigma \times V^{N\ell})} \cong \mathbb{K}_{\pi_z({}^d\mathcal{W}_\ell^N \cap (\Sigma \times V^{N\ell}))} = \mathbb{K}_{\mathcal{W}_\ell^N(\Sigma)}.$$

Therefore, $\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_\Sigma\right) = \mathrm{R}\pi_{\underline{\mathbf{q}}!}(\mathbb{K}_{\mathcal{W}_\ell^N(\Sigma)}) \cong \mathrm{R}\Gamma_c\left(\mathcal{W}_\ell^N(\Sigma), \mathbb{K}\right).$

The statements involve $\Sigma = \Omega_T^\circ$ follow from the discussion above the lemma. □

Remark 3.15. The condition in the lemma is true for compact convex sets Σ. For our last applications, we need to, in adition, consider γ-closed sets for $\gamma = (-\infty, 0]^d$. For a closed set $\Sigma \subset \gamma$, the γ-closure is defined as $\Sigma_\gamma = (\Sigma - \gamma) \cap \gamma$. We say Σ is γ-*closed* if $\Sigma_\gamma = \Sigma$. For example, $\gamma \setminus (\mathring{\gamma} + z)$ is γ-closed for $z \in \gamma$, and the intersection of two γ-closed sets is γ-closed. The γ-closed sets satisfy the condition of Lemma 3.14. Indeed, for a closed cube $[x, y]$ with $x, y \in \gamma$, a γ-closed Σ, and any $z \in \Sigma \cap [x, y]$, we have $\overline{xz} \subset \Sigma \cap [x, y]$. Therefore, $\Sigma \cap [x, y]$ is star-shaped and then contractible.

As $z \mapsto \mathcal{W}(z)$ is a decreasing map, one can see that $\mathcal{W}_\ell^N(\Sigma) = \mathcal{W}_\ell^N(\Sigma_\gamma)$ for all compact subset $\Sigma \subset \gamma$. In particular, if Σ satisfies the condition of Lemma 3.14, then the lemma implies that

$$\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\Sigma_\gamma}\right) \xrightarrow{\cong} \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_{\Sigma}\right),$$

which can be seen as a generalization of (3.25) (which is only true for compact convex subsets) for compact sets satisfying the condition of Lemma 3.14. Later, we will mainly focus on γ-closed sets Σ.

Now, to understand the cohomology of $\mathcal{W}_\ell^N(\Sigma)$ (see (3.29)), we start from a special case that Σ is an "almost closed cube", which will be defined in Lemma 3.16. Let us recall some notation and introduce some new ones.

First, recall that, for $x, y \in \mathbb{R}^d$, we let $(x, y] = \prod_{i=1}^d (x_i, y_i]$ be the half-open cube from x to y. Similarly, we define open cubes and closed cubes in this way. Also, recall $O \in \mathbb{R}^d$ is the origin, and we set $\mathbb{1} = (1, \dots, 1)$ and $e_i = (\delta_{ij})_{j=1}^d$ where δ_{ij} stands for the Kronecker symbol. For simplicity, we also denote $C_x = [x, x + \mathbb{1})$ for $x \in \mathbb{R}^d$.

Next, for a compact γ-closed set $\Sigma \subset \gamma$, we set

$$\begin{aligned} J_\Sigma &= (-\Sigma) \cap \mathbb{Z}_{\geq 0}^d = \{v \in \mathbb{Z}_{\geq 0}^d : (-\Sigma) \cap C_v \neq \varnothing\}, \\ \partial J_\Sigma &= \{v \in J_\Sigma : \forall i, v + e_i \notin J_\Sigma\} = \{v \in J_\Sigma : (-\Sigma) \cap (\overline{C_v} \setminus C_v) = \varnothing\}. \end{aligned}$$

The compactness of Σ shows that both J_Σ and ∂J_Σ are finite sets.

Lemma 3.16. *Let $\Sigma \subset \gamma$ be a compact γ-closed set. Then $\partial J_\Sigma = \{v\}$ for some $v \in \mathbb{Z}_{\geq 0}^d$ if and only if $[O, v] \subset -\Sigma \subset [O, v + \mathbb{1})$ for the same $v \in \mathbb{Z}_{\geq 0}^d$.*

We say that Σ is an almost cube *if it satisfies these equivalent conditions.*

Proof. When $[O, v] \subset -\Sigma \subset [O, v + \mathbb{1})$, taking the intersection with $\overline{C_w} \setminus C_w$ for all $w \in J_\Sigma$, we obtain

$$[O, v] \cap (\overline{C_w} \setminus C_w) \subset (-\Sigma) \cap (\overline{C_w} \setminus C_w) \subset [O, v + \mathbb{1}) \cap (\overline{C_w} \setminus C_w).$$

Then we can obtain $\partial J_\Sigma = \{v\}$ from that $[O, v] \cap (\overline{C_w} \setminus C_w) = \varnothing$ only when $v = w$.

Conversely, when $\partial J_\Sigma = \{v\}$, we have $-v \in \Sigma$. So $\Sigma = \Sigma_\gamma$ implies $[-v, O] = \{-v\}_\gamma \subset \Sigma$. Now, suppose $-\Sigma \not\subset [O, v + \mathbb{1})$, then there is a $z \in \Sigma$ such that $-z_i = v_i + 1$ for some $i \in [d]$. Therefore, $v + e_i \in J_\Sigma$. If $v + e_i \notin \partial J_\Sigma$, the argument repeats and there exists another $j \in [d]$ such that

$v + e_i + e_j \in J_\Sigma$. We can continue until we obtain a index set I (with possible multiplicities) such that $v + \sum_I e_i \in \partial J_\Sigma$. Since J_Σ is a finite set, then the index set must be finite. However, $\partial J_\Sigma = \{v\}$, then $v + \sum_I e_i \notin \partial J_\Sigma$. Hence we get a contradiction. Then $-\Sigma \subset [O, v + \mathbb{1})$. $\square$

Here, we are going to prove a refinement of the fact that ${}^d\mathcal{W}_\ell^N(-v)$ is properly homotopic to $\mathbb{R}^{d+2I(-v)}$ as noticed before Lemma 3.12.

Lemma 3.17. *For a compact γ-closed set $\Sigma \subset \gamma$ with $\partial J_\Sigma = \{v\}$, i.e. Σ is an almost cube, the subspace $\mathbb{R}^d \times \mathbb{C}^{I(-v)}$ is a strong deformation retract of $\mathcal{W}_\ell^N(\Sigma)$ under a proper deformation retraction. Moreover, $\Delta_{V^{N\ell}} \cong \mathbb{R}^d \times \{0\} \subset \mathbb{R}^d \times \mathbb{C}^{I(-v)}$ is invariant under the retraction.*

Proof. Here, we use the diagonal form of Q_z we introduced in (3.17). Then the coordinate system on $\left(\mathbb{R} \times \mathbb{C}^{\frac{N\ell-1}{2}}\right)^d$ is $(x_k^i)_{i,k} = (\mathbf{x}_k)_k = (\mathbf{x}^i)_i$ with $\mathbf{x}_k = (x_k^1, \dots, x_k^d)$ and $\mathbf{x}^i = (x_0^i, \dots, x_k^i)$, where $i \in [d] = \{1, \dots, d\}$, $k = 0, 1, \dots (N\ell - 1)/2$ if ℓ is odd and $k = 0, 1, \dots N\ell/2$ if ℓ is even. For shortness, we only deal with the ℓ odd case. The ℓ even case has the same proof with minor corrections on the notation. Recall that

$$\begin{aligned} \mathcal{W}_\ell^N(\Sigma) &= \left\{(\mathbf{x}^i)_i = (x_0^i, x_k^i)_{i,k} : \exists z \in \Sigma, \forall i,\ Q_{z_i}(\mathbf{x}^i) \geq 0\right\}, \\ \Delta_{V^{N\ell}} &= \left\{(\mathbf{x}^i)_i = (x_0^i, x_k^i)_{i,k} : \forall k \geq 1, i \in [d],\ \text{such that } x_k^i = 0\right\}. \end{aligned}$$

For $0 \leq m \leq (N\ell - 1)/2$, $i \in [d]$, consider $h_m^i : \mathbb{R} \times \mathbb{C}^{\frac{N\ell-1}{2}} \times [0,1] \to \mathbb{R} \times \mathbb{C}^{\frac{N\ell-1}{2}}$,

$$h_m^i(x_0^i, x_+^i, x_-^i, t) = h_{m,t}^i(x_0^i, x_+^i, x_-^i) = (x_0^i, x_+^i, t x_-^i),$$

where $x_+^i = (x_1^i, \dots, x_m^i)$, $x_-^i = (x_{m+1}^i, x_{m+2}^i, \dots, x_{(N\ell-1)/2}^i)$.

By assumption of the lemma, $v \in -\Sigma \subset [0, N\ell/4)^d$. Then we have $0 \leq v_i < N\ell/4 \leq (N\ell - 1)/2$. Now, define $H_v : \left(\mathbb{R} \times \mathbb{C}^{\frac{N\ell-1}{2}}\right)^d \times [0,1] \to \left(\mathbb{R} \times \mathbb{C}^{\frac{N\ell-1}{2}}\right)^d$ by

$$H_{v,t} = h_{v_1,t}^1 \times \cdots \times h_{v_d,t}^d.$$

Then we have $H_{v,1}$ is the identity map. Next, we have the following:

- $H_{v,t}(\mathcal{W}_\ell^N(\Sigma)) \subset \mathcal{W}_\ell^N(\Sigma)$. Indeed, $(\mathbf{x}^i)_i \in \mathcal{W}_\ell^N(\Sigma)$ implies there exists $z \in \Sigma$ such that for all $i \in [d]$, we have $Q_{z_i}(\mathbf{x}^i) \geq 0$. So, in the diagonal form

(3.18), we have for all $i \in [d]$,

$$\lambda_0(z_i)|x_0^i|^2 + 2\sum_{k=1}^{v_i} \lambda_k(z_i)|x_k^i|^2 \geq 2 \sum_{k \geq v_i+1} (-\lambda_k(z_i))|x_k^i|^2.$$

Now $-\Sigma \subset [O, v + \mathbb{1})$ implies that $z_i < v_i + 1$ for all $i \in [d]$, hence $\lambda_k(z_i) < 0$ for $k \geq v_i + 1$ and for all $i \in [d]$. So

$$\begin{aligned}\lambda_0(z_i)|x_0^i|^2 + 2\sum_{k=1}^{v_i} \lambda_k(z_i)|x_k^i|^2 &\geq 2 \sum_{k \geq v_i+1} (-\lambda_k(z_i))|x_k^i|^2 \\ &\geq 2t^2 \sum_{k \geq v_i+1} (-\lambda_k(z_i))|x_k^i|^2,\end{aligned}$$

i.e., $Q_{z_i}(h_{v_i,t}(\mathbf{x}^i)) \geq 0$ for all $i \in [d]$. Hence $H_{v,t}(\mathbf{x}^1, \ldots, \mathbf{x}^d) \in \mathcal{W}_\ell^N(\Sigma)$.

- $H_v|_{\mathcal{W}_\ell^N(\Sigma)}$ is proper. Indeed, taking $(\mathbf{x}^i)_i \in \mathcal{W}_\ell^N(\Sigma)$ such that $H_v((\mathbf{x}^i)_i) \in [-M, M]^d$, we have that $\sum_{k=0}^{v_i} |x_k^i|^2 + \sum_{k \geq v_i+1} |tx_k^i|^2 \leq M$, for all $i \in [d]$.

  Obviously, $\sum_{k=0}^{v_i} |x_k^i|^2 \leq M$, for all $i \in [d]$, and

$$\begin{aligned}2 \max_{\substack{k=0,\ldots,v_i \\ z\in\Sigma}} |\lambda_k(z_i)| M &\geq \lambda_0(z_i)|x_0^i|^2 + 2\sum_{k=0}^{v_i} \lambda_k(z_i)|x_k^i|^2 \\ &\geq 2 \sum_{k \geq v_i+1} (-\lambda_k(z_i))|x_k^i|^2 \\ &\geq 2 \min_{\substack{k \geq v_i+1 \\ z\in\Sigma}} |\lambda_k(z_i)| \sum_{k \geq v_i+1} |x_k^i|^2.\end{aligned}$$

Since $\lambda_k(z_i) < 0$ for $k \geq v_i + 1$, and $z \in \Sigma$, we have $\min_{\substack{k \geq v_i+1 \\ z\in\Sigma}} |\lambda_k(z_i)| > 0$. Consequently,

$$\sum_{k \geq v_i+1} |x_k^i|^2 \leq \frac{\max\limits_{\substack{k=1,\ldots,v_i \\ z\in\Sigma}} |\lambda_k(z_i)|}{\min\limits_{\substack{k \geq v_i+1 \\ z\in\Sigma}} |\lambda_k(z_i)|} M =: KM.$$

It means that $\sum_{k=0}^{v_i} |x_k^i|^2 + \sum_{k \geq v_i+1} |x_k^i|^2 \leq (1 + K)M$, for all $i \in [d]$, where $K = K(\Sigma)$ is a constant only depending on $\mathcal{W}_\ell^N(\Sigma)$.

So, we have shown that the pre-image of a bounded set under $H_v|_{\mathcal{W}_\ell^N(\Sigma)}$ is bounded. It means that $H_v|_{\mathcal{W}_\ell^N(\Sigma)}$ is proper.

Hence $H_v|_{\mathcal{W}_\ell^N(\Sigma)}$ is a proper homotopy with $H_{v,1}|_{\mathcal{W}_\ell^N(\Sigma)} = \mathrm{Id}_{\mathcal{W}_\ell^N(\Sigma)}$

• $\mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\} \subset \mathcal{W}_\ell^N(\Sigma)$. Let $(\mathbf{x}^i)_i \in \mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\}$. This means that for all $i \in [d]$, $\mathbf{x}^i = (x_0^i, x_+^i, x_-^i)$ satisfies $x_-^i = 0$. Since $z = -v \in \Sigma$, by assumption, it is enough to check that $Q_{-v_i}(\mathbf{x}^i) \geq 0$ for all $i \in [d]$. Now $\lambda_k(-v_i) \geq 0$ for $k = 0, 1, \ldots, v_i$. Then for all $i \in [d]$, we have

$$Q_{-v_i}(x_0^i, x_+^i, 0) = \lambda_0(-v_i)|x_0^i|^2 + 2\sum_{k=1}^{v_i} \lambda_k(-v_i)|x_k^i|^2 \geq 0.$$

So $(\mathbf{x}^i)_i \in \mathcal{W}_\ell^N(\Sigma)$ and then $\mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\} \subset \mathcal{W}_\ell^N(\Sigma)$.

• $H_{v,0}(\mathcal{W}_\ell^N(\Sigma)) \subset \mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\}$. Indeed for $(\mathbf{x}^i)_i \in \mathcal{W}_\ell^N(\Sigma)$, we have $h_{v_i,0}^i(\mathbf{x}^i) = (x_0^i, x_+^i, 0)$ for all $i \in [d]$. Then $H_{v,0}(\mathcal{W}_\ell^N(\Sigma)) \subset \mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\}$.

On the other hand, by definition of $h_{v_i}^i$, we have

$$h_{v_i}^i(x_0^i, x_+^i, 0, t) = (x_0^i, x_+^i, t0) = (x_0^i, x_+^i, 0).$$

So $H_{v,t}|_{\mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\}} = \mathrm{Id}_{\mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\}}$ for all $t \in [0,1]$. Therefore, $\mathbb{R}^d \times \mathbb{C}^{I(-v)} \times \{0\}$ is a proper strong deformation retract of $\mathcal{W}_\ell^N(\Sigma)$ under $H_{v,t}|_{\mathcal{W}_\ell^N(\Sigma)}$. ☐

Below, we will frequently use the equivariant global section functor $\mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_W\right)$ for locally closed set $W \subset \gamma$. Then we denote it by

$$\Gamma\mathcal{E}(W) \coloneqq \mathrm{R}\Gamma_c\left(\mathbb{R}^d, (\mathcal{E}_\ell^{\overset{L}{\boxtimes} d})_W\right), \tag{3.31}$$

for shortening the length of notation until the end of the subsection.

Lemma 3.18. *Let $\Sigma \subset \gamma$ be a compact γ-closed set such that $\Sigma \subset (-p_\ell \mathbb{1}, O]$. Recall the notation $I(\Sigma)$ at* (3.10)*. Then*

$$\mathrm{Ext}_{\mathbb{Z}/\ell}^{q-d}(\Gamma\mathcal{E}(\Sigma \setminus O), \mathbb{F}_{p_\ell}) \cong 0 \qquad \text{if } q \notin [-2I(\Sigma), -1].$$

The $\mathbb{F}_{p_\ell}$-vector space $\mathrm{Ext}_{\mathbb{Z}/\ell}^{q-d}(\Gamma\mathcal{E}(\Sigma \setminus O), \mathbb{F}_{p_\ell})$ is finite dimensional.

Proof. We proceed by induction on $|J_\Sigma|$. We notice that the maximum $I(\Sigma)$ can be achieved by some v since $\Sigma \cap \mathbb{Z}^d$ is finite. Moreover, if $v \in J_\Sigma$ satisfies

$I(-v) = I(\Sigma)$, then $v \in \partial J_\Sigma$. We will use the excision distinguished triangle

$$\Gamma\mathcal{E}(\Sigma \setminus O) \to \Gamma\mathcal{E}(\Sigma) \xrightarrow{\eta_\Sigma} \Gamma\mathcal{E}(O) \xrightarrow{+1} . \tag{3.32}$$

If $|J_\Sigma| = 1$, that is $J_\Sigma = \{0\}$, then Lemma 3.17 shows that η_Σ is an isomorphism in the derived category. Then $\Gamma\mathcal{E}(\Sigma \setminus O) \cong 0$ by (3.32) and the result follows.

Now, we suppose the result is true for all Σ' such that $|J_{\Sigma'}| < |J_\Sigma|$ and we distinguish the cases $|\partial J_\Sigma| = 1$ and $|\partial J_\Sigma| > 1$.

1) If $\partial J_\Sigma = \{v\}$ is a singleton, i.e. Σ is an almost cube. The case $v = 0$ is already done and we assume $I(-v) > 0$. Then the excision sequence (3.32), Lemma 3.14 and Lemma 3.17 together show the isomorphisms in the equivariant derived category

$$\Gamma\mathcal{E}(\Sigma \setminus O) \cong \mathrm{R}\Gamma_c(\mathcal{W}_\ell^N(\Sigma) \setminus \Delta_{V^{N\ell}}, \mathbb{F}_{p_\ell}) \cong \mathrm{R}\Gamma(S^{2I(-v)-1}, \mathbb{F}_{p_\ell})[-d-1],$$

where the action of $\mathbb{Z}/\ell$ on $S^{2I(-v)-1}$ is given in (3.20). We have

$$\begin{aligned}
\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(\Sigma \setminus O), \mathbb{F}_{p_\ell}) &\cong \mathrm{Ext}^{*+1}_{\mathbb{Z}/\ell}\left(\mathrm{R}\Gamma(S^{2I(-v)-1}, \mathbb{F}_{p_\ell}), \mathbb{F}_{p_\ell}\right) \\
&\cong \mathrm{Ext}^{*+1}\left((\mathbb{F}_{p_\ell})_{S^{2I(-v)-1}}, \omega^!_{S^{2I(-v)-1}}\right) \\
&\cong H^{\mathbb{Z}/\ell}_{-*-1}(S^{2I(-v)-1}, \mathbb{F}_{p_\ell}),
\end{aligned}$$

where we used the equivariant Poincaré duality, which holds since $S^{2I(-v)-1}$ is compact and orientable.

Under the assumption $\Sigma \subset (-p_\ell \mathbb{1}, O]$, the $\mathbb{Z}/\ell$-action is free by (3.20). Hence $H^{\mathbb{Z}/\ell}_{-*-1}(S^{2I(-v)-1}, \mathbb{F}_{p_\ell})$ computes the usual cohomology of the quotient $Q_{\mathbb{Z}/\ell} = S^{2I(-v)-1}/(\mathbb{Z}/\ell)$, which is the lens space of dimension $2I(-v) - 1$.

Then, we have

$$H_q(Q_{\mathbb{Z}/\ell}) = \begin{cases} \mathbb{F}_{p_\ell}, & q \in [0, 2I(-v) - 1], \\ 0, & q \notin [0, 2I(-v) - 1]. \end{cases}$$

Converting to cohomology degree, we obtain: $\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(\Sigma \setminus O), \mathbb{F}_{p_\ell})$ is concentrated in $[-2I(-v), -1]$ and finite dimensional.

The proof of this part is independent of our induction, so it can be applied to the second case.

2) If $|\partial J_\Sigma| \geq 2$, take $v \in \partial J_\Sigma$ such that $I(-v) = I(\Sigma)$. Then we can take $1 > \delta > 0$ such that $\Sigma \cap (\gamma + (\delta \mathbb{1} - v)) \subset (-v, 0]$. This is possible due to the

compactness of Σ. Let us define:

$$A = [\Sigma \cap (\gamma + (\delta\mathbb{1} - v))]_\gamma,$$
$$B = \Sigma \cap [\gamma \setminus (\mathring{\gamma} + (\delta\mathbb{1} - v))]. \tag{3.33}$$

Then we have a closed covering $\Sigma = A \cup B$. Moreover, both A and B are compact γ-closed sets, then so is $A \cap B$ (see Figure 1).

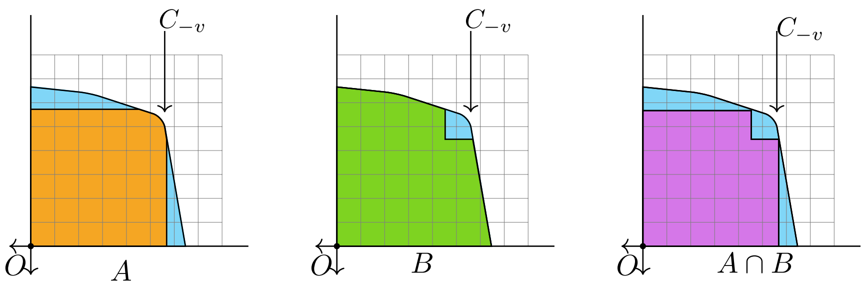

Figure 1: The picture illustrate the construction of A, B. Σ is the background blue set.

Then we have the Mayer-Vietoris triangle,

$$\Gamma\mathcal{E}(\Sigma \setminus O) \to \Gamma\mathcal{E}(A \setminus O) \oplus \Gamma\mathcal{E}(B \setminus O) \to \Gamma\mathcal{E}((A \cap B) \setminus O) \xrightarrow{+1} .$$

Next, we apply the $\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(-, \mathbb{F}_{p_\ell}) \cong \mathrm{Ext}^{*}_{\mathbb{Z}/\ell}(-, \mathbb{F}_{p_\ell}[-d])$ to obtain a long exact sequence

$$\begin{aligned} &\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}((A \cap B) \setminus O), \mathbb{F}_{p_\ell}) \to \\ &\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(A \setminus O), \mathbb{F}_{p_\ell}) \oplus \mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(B \setminus O), \mathbb{F}_{p_\ell}) \to \\ &\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(\Sigma \setminus O), \mathbb{F}_{p_\ell}) \xrightarrow{+1} . \end{aligned} \tag{3.34}$$

By our construction (3.33), we have

- $|\partial J_A| = 1$. Hence we can apply the result of (1). So that $\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(A \setminus O), \mathbb{F}_{p_\ell})$ is concentrated in $[-2I(A), -1] \subset [-2I(\Sigma), -1]$.
- $|J_B| < |J_\Sigma|$. We can use the induction hypothesis, hence $\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(B \setminus O), \mathbb{F}_{p_\ell})$ is concentrated in $[-2I(B), -1] \subset [-2I(\Sigma), -1]$.
- $|J_{A\cap B}| < |J_\Sigma|$, since $J_{A\cap B} \subset J_A$ but $v \notin J_{A\cap B}$. Then we can use the induction hypothesis, that $\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}((A \cap B) \setminus O), \mathbb{F}_{p_\ell})$ is concentrated in $[-2I(A \cap B), -1]$. Moreover, in J_A, v is the only lattice point such that

$I(-v) = I(\Sigma)$, then for all $v' \in J_{A\cap B} \subset J_A \setminus \{v\}$, we have $I(-v') < I(\Sigma)$. Then $|I(A\cap B)| < I(\Sigma)$, and $[-2I(A\cap B), -1] \subset [-2I(\Sigma)+2, -1]$.

Therefore, it follows from the long exact sequence (3.34) that $\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(\Sigma\setminus O), \mathbb{F}_{p_\ell})$ is concentrated in $[-2I(\Sigma), -1]$ and finite dimensional. □

Now, we are in the position to prove Theorem 3.6.

Proof of Theorem 3.6. The equation (3.30) says that $(F_\ell(X_\Omega, \mathbb{F}_{p_\ell}))_T \cong \Gamma\mathcal{E}(\Omega_T^\circ)$. Now, consider the inclusion sequence $\{O\} \subset \overline{ZO} \subset \Omega_T^\circ$ of closed sets. Then we have a commutative diagram:

$$\begin{array}{ccccc} \Gamma\mathcal{E}(\Omega_T^\circ) & \longrightarrow & \Gamma\mathcal{E}(\overline{ZO}) & \xrightarrow{\cong} & \mathbb{F}_{p_\ell}[-d-2I(Z)] \\ & \searrow & \downarrow & & \downarrow{\scriptstyle k_Z u^{I(Z)}} \\ & & \Gamma\mathcal{E}(O) & \xrightarrow{\cong} & \mathbb{F}_{p_\ell}[-d] \end{array}$$

By definition, the inclined arrow compose with the bottom isomorphism gives the fundamental class $\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$. The terms $\Gamma\mathcal{E}(\overline{ZO})$ and $\Gamma\mathcal{E}(O)$ are computed using Lemma 3.12 and (3.25). Lemma 3.12 also shows that the vertical morphism is $k_Z u^{I(Z)}$, where k_Z is a constant only depends on Z. We absorb the constant into the horizontal arrow, then we call it $\Lambda_{Z,\ell}$. Therefore, the commutative diagram induces a decomposition $\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}) = u^{I(Z)}\Lambda_{Z,\ell}$.

Now, let us embed the fundamental class $\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})$ into the excision triangle (the triangle (3.32) for $\Sigma = \Omega_T^\circ$)

$$\Gamma\mathcal{E}(\Omega_T^\circ\setminus O) \to \Gamma\mathcal{E}(\Omega_T^\circ) \xrightarrow{\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell})} \Gamma\mathcal{E}(O) \xrightarrow{+1} . \tag{3.35}$$

So, after applying $\mathrm{RHom}_{\mathbb{Z}/\ell}(-, \mathbb{F}_{p_\ell}[-d])$, we get the distinguished triangle:

$$\begin{aligned} &\mathrm{R}\Gamma(V, \omega_V^{\mathbb{Z}/\ell}) \to C_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}) \xrightarrow{\mathrm{RHom}_{\mathbb{Z}/\ell}(\eta_T^{\mathbb{Z}/\ell}(X_\Omega, \mathbb{F}_{p_\ell}), \mathbb{F}_{p_\ell}[-d])} \\ &\mathrm{RHom}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(\Omega_T^\circ\setminus O), \mathbb{F}_{p_\ell}[-d]) \xrightarrow{+1} . \end{aligned} \tag{3.36}$$

Here $V \cong \mathbb{R}^d$ and it is equipped with the trivial $\mathbb{Z}/\ell$-action. Taking cohomology for the distinguished triangle, we get a long exact sequence of the

Chiu-Tamarkin cohomology:

$$\begin{aligned}(3.37)\qquad & H^*_{\mathbb{Z}/\ell}(V,\mathbb{F}_{p_\ell}) \to H^*C_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell}) \xrightarrow{\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\eta_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell}),\mathbb{F}_{p_\ell})} \\ & \mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(\Omega_T^\circ \setminus O),\mathbb{F}_{p_\ell}) \xrightarrow{+1} .\end{aligned}$$

When $0 \le T < p_\ell/\|\Omega_1^\circ\|_\infty$, we have $\Omega_T^\circ \subset (-p_\ell \mathbb{1}, O]$.

Then we can apply Lemma 3.18, $\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(\Omega_T^\circ \setminus O),\mathbb{F}_{p_\ell})$ is a finite dimensional graded $\mathbb{F}_{p_\ell}$ vector space which is concentrated in degrees $[-2I(\Omega_T^\circ),-1]$. Then, it is torsion as a $\mathbb{F}_{p_\ell}[u]$-module.

One the other hand, $H^*_{\mathbb{Z}/\ell}(V,\mathbb{F}_{p_\ell}) \cong \mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{F}_{p_\ell}[-d],\mathbb{F}_{p_\ell}[-d]) \cong \mathbb{F}_{p_\ell}[u,\theta]$ is concentrated in $[0,\infty)$.

Therefore, after tensoring with $\mathbb{F}_{p_\ell}((u))$,

$$\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\eta_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell}),\mathbb{F}_{p_\ell}) \otimes_{\mathbb{F}_{p_\ell}[u]} \mathbb{F}_{p_\ell}((u))$$

is an isomorphism of $\mathbb{F}_{p_\ell}((u))$-vector spaces. Then, we conclude that $\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\eta_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell}),\mathbb{F}_{p_\ell}) \neq 0$ and so $\eta_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell}) \neq 0$. Moreover, we have that $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell})$ is a finitely generated $\mathbb{F}_{p_\ell}[u]$ module whose rank is 2 and the torsion part of $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell})$ is $\mathrm{Ext}^{*-d}_{\mathbb{Z}/\ell}(\Gamma\mathcal{E}(\Omega_T^\circ \setminus O),\mathbb{F}_{p_\ell})$.

Now, for the $\mathbb{F}_{p_\ell}[u]$ module $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell})$, its free part is concentrated in $[0,\infty)$ and its torsion part are concentrated in degrees $[-2I(\Omega_T^\circ),-1]$. Then the minimal degree of $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell})$ is at least $-2I(\Omega_T^\circ)$, and torsion elements of $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell})$ only appear in degree $[-2I(\Omega_T^\circ),-1]$.

On the other hand, this estimate is sharp. Indeed, we take $Z \in \Omega_T^\circ$ such that $I(Z) = I(\Omega_T^\circ)$. Then the decomposition $\eta_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell}) = u^{I(Z)}\Lambda_{Z,\ell}$ shows that we have a degree equation: $0 = 2|I(Z)| + |\Lambda_{Z,\ell}|$. Then $|\Lambda_{Z,\ell}| = -2I(\Omega_T^\circ)$ realizes the minimal degree $-2I(\Omega_T^\circ)$.

For the ellipsoid $E(a) = X_{\Omega_{E(a)}}$ (see Example 3.4-(2)), let $Z = (-T/a_1,\dots,-T/a_d)$, then $\left(\Omega_{E(a)}\right)_T^\circ = \overline{ZO}$ is a segment. So, we can compute $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell})$ directly from Lemma 3.12, and $\Lambda_{Z,\ell}$ we defined above induces an isomorphism of A module. So $H^*C_T^{\mathbb{Z}/\ell}(X_\Omega,\mathbb{F}_{p_\ell})$ is torsion free as a $\mathbb{F}_{p_\ell}[u]$-module. □

4. Contact invariants

I will explain how the Chiu-Tamarkin complex works for the contact geometry of (contact) admissible open sets in the prequantized space $T^*X \times S^1$.

For any open set $\mathscr{U} \subset T^*X \times S^1$, we can lift it to a $\mathbb{Z}$-invariant set $\widetilde{\mathscr{U}} \subset J^1X$ in the sense $T'_k(\widetilde{\mathscr{U}}) = \widetilde{\mathscr{U}}$, where $T'_k(\mathbf{q}, \mathbf{p}, t) = (\mathbf{q}, \mathbf{p}, t + k)$ for $k \in \mathbb{Z}$. In this way, we can discuss sheaves microsupported in $J^1X \setminus \widetilde{\mathscr{U}}$. Then $\mathcal{D}_{J^1X\setminus\widetilde{\mathscr{U}}}(X)$ and its left semi-orthogonal complement are all well-formulated. Specifically, for $\mathscr{Z} = J^1X \setminus \widetilde{\mathscr{U}}$, we define

$$\begin{aligned}
\mathcal{D}^c_{\mathscr{Z}}(X) =& \{F \in \mathcal{D}(X) : \mu s_L(F) \subset \mathscr{Z}\},\\
\mathcal{D}^c_{\mathscr{U}}(X) =& {}^{\perp}\mathcal{D}^c_{\mathscr{Z}}(X), \text{ the left orthogonal complement of } \mathcal{D}^c_{\mathscr{Z}}(X).
\end{aligned}$$

Same as the symplectic case, we can define the notion of admissibility and microlocal kernels. To make it compatible with the Hamiltonian action of contact isotopy as we discussed in subsection 1.3, we will use composition functors rather than convolution functors. On the other hand, in the symplectic case, we require that microlocal kernels are objects in the Tamarkin category. Now, we need a (2-variable) variant version of the Tamarkin category for contact microlocal kernels. Let $\mathscr{D}(X^2)$ be the full triangulated subcategory $\{F \in D(X^2 \times \mathbb{R}^2) : F \circ \mathbb{K}_{\{t_2 \geq t_1\}} \xrightarrow{\cong} F\}$ of $D(X^2 \times \mathbb{R}^2)$. Then we define

Definition 4.1. We say $\mathscr{U}$ is $\mathbb{K}$-*admissible* if there is a distinguished triangle

$$\mathscr{P}_{\mathscr{U}} \to \mathbb{K}_{\Delta_{X^2}\times\{t_2\geq t_1\}} \to \mathscr{Q}_{\mathscr{U}} \xrightarrow{+1},$$

in $\mathscr{D}(X^2)$ such that the composition functor $\circ\mathscr{P}_{\mathscr{U}}$ is right adjoint to $\mathcal{D}^c_{\mathscr{U}}(X) \hookrightarrow \mathcal{D}(X)$ and $\circ\mathscr{Q}_{\mathscr{U}}$ is left adjoint to $\mathcal{D}^c_{\mathscr{Z}}(X) \hookrightarrow \mathcal{D}(X)$, i.e.,

$$\mathcal{D}^c_{\mathscr{Z}}(X) \xleftarrow{\circ\mathscr{Q}_{\mathscr{U}}} \mathcal{D}(X) \xrightarrow{\circ\mathscr{P}_{\mathscr{U}}} \mathcal{D}^c_{\mathscr{U}}(X),$$

are two microlocal projectors.

Such a pair of sheaves $(\mathscr{P}_{\mathscr{U}}, \mathscr{Q}_{\mathscr{U}})$ together with the distinguished triangle give an orthogonal decomposition of $\mathcal{D}(X)$ by Proposition 1.10. We call the pair $(\mathscr{P}_{\mathscr{U}}, \mathscr{Q}_{\mathscr{U}})$ *microlocal kernels* associated with $\mathscr{U} \subset T^*X \times S^1$, and the distinguished triangle as the defining triangle of $\mathscr{U}$.

We say $\mathscr{U}$ is *admissible* if $\mathscr{U}$ is $\mathbb{Z}$-admissible.

The uniqueness and functoriality has the same proof, just need to replace convolution by composition. We have the existence of kernels for the prequantized open set $U \times S^1$ where $U \subset T^*X$ is a symplectic admissible open set. Precisely, we have the following proposition.

Proposition 4.2. *If $U \subset T^*X$ is (symplectic) admissible by the following distinguished triangle:*

$$P_U \to \mathbb{K}_{\Delta_{X^2} \times \{t \geq 0\}} \to Q_U \xrightarrow{+1} .$$

Then $U \times S^1 \subset T^*X \times S^1$ is (contact) admissible by the following distinguished triangle:

$$\mathscr{P}_{U \times S^1} \to \mathbb{K}_{\Delta_{X^2} \times \{t_2 \geq t_1\}} \to \mathscr{Q}_{U \times S^1} \xrightarrow{+1},$$

where $\mathscr{P}_{U \times S^1} = m^{-1} P_U$, $\mathscr{Q}_{U \times S^1} = m^{-1} Q_U$ and $m(t_1, t_2) = t_2 - t_1$.

Notice that we have $\mathbb{K}_{\Delta_{X^2} \times \{t_2 \geq t_1\}} = m^{-1} \mathbb{K}_{\Delta_{X^2} \times [0,\infty)}$.

Proof. The second distinguished triangle comes from applying m^{-1} to the first one and we have $m^{-1}F \in \mathscr{D}(X^2)$ for $F \in \mathcal{D}(X^2)$. On the other hand, as we mentioned in (1) of Remark 1.9, we have

$$F \star P_U \cong F \circ \mathscr{P}_{U \times S^1}, \quad F \star Q_U \cong F \circ \mathscr{Q}_{U \times S^1},$$

for $F \in \mathcal{D}(X)$. Finally, as $\widetilde{U \times S^1} = U \times \mathbb{R}$, we have that $\mu s_L(F) \subset J^1X \setminus \widetilde{U \times S^1}$ if and only if $\mu s(F) \subset T^*X \setminus U$. Then the result follows. □

Now, we can define the contact Chiu-Tamarkin complex for admissible open sets $\mathscr{U} \subset T^*X \times S^1$. As in the symplectic case, let us introduce the adjoint pair first:

$$F \in D_{\mathbb{Z}/\ell}((X^2 \times \mathbb{R}_t^2)^\ell) \underset{\beta^c_{\ell,T,X}}{\overset{\alpha^c_{\ell,T,X}}{\rightleftarrows}} D_{\mathbb{Z}/\ell}(\mathrm{pt}) \ni G,$$

defined by:

$$\begin{aligned} \alpha^c_{\ell,n,X}(F) &= (i_n^\ell)^{-1} \mathrm{R}\pi_{\underline{\mathbf{q}}!} \tilde{\Delta}_X^{-1} \mathrm{R}\widetilde{m}_! (F), \\ \beta^c_{\ell,n,X}(G) &= \widetilde{m}^! \tilde{\Delta}_{X*} \pi^!_{\underline{\mathbf{q}}} i^\ell_{n*} G[-1], \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} &\widetilde{m} : (X^2 \times \mathbb{R}^2)^\ell \to X^{2\ell} \times \mathbb{R}^\ell, \\ &\widetilde{m}(\underline{\mathbf{q}}, t_1^1, t_1^2, \ldots, t_\ell^1, t_\ell^2) = (\underline{\mathbf{q}}, t_\ell^2 - t_1^1, t_1^2 - t_2^1, \ldots, t_{\ell-1}^2 - t_\ell^1); \\ &\widetilde{\Delta}_X : X^\ell \times \mathbb{R}^\ell \to X^{2\ell} \times \mathbb{R}^\ell, \\ &\widetilde{\Delta}_X(\mathbf{q}_1, \ldots, \mathbf{q}_\ell, \underline{t}) = (\mathbf{q}_\ell, \mathbf{q}_1, \mathbf{q}_1, \ldots, \mathbf{q}_{\ell-1}, \mathbf{q}_{\ell-1}, \mathbf{q}_n, \underline{t}); \end{aligned}$$

$$\pi_{\underline{\mathbf{q}}} : X^\ell \times \mathbb{R}^\ell \to \mathbb{R}^\ell;$$
$$i^\ell_n(\mathrm{pt}) = (n, \ldots, n) \in \mathbb{R}^\ell,$$

where $\underline{\mathbf{q}} = (\mathbf{q}_1, \ldots, \mathbf{q}_\ell)$ and $\underline{t} = (t_1, \ldots, t_\ell)$.

Definition 4.3. With the notation above, for $\ell \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define the *contact Chiu-Tamarkin complex* as follows:

$$\begin{aligned}\mathscr{C}^{\mathbb{Z}/\ell}_{n\ell}(\mathscr{U}, \mathbb{K}) &= \mathrm{RHom}_{\mathbb{Z}/\ell}\left(\alpha^c_{\ell,n,X}(\mathscr{P}^{\overset{L}{\boxtimes}\ell}_{\mathscr{U}}), \mathbb{K}[-d]\right)\\ &\cong \mathrm{RHom}_{\mathbb{Z}/\ell}\left(\mathscr{P}^{\overset{L}{\boxtimes}\ell}_{\mathscr{U}}, \beta^c_{\ell,n,X}\mathbb{K}[-d]\right).\end{aligned}$$

Compare to the symplectic case, the parameter T is replaced by a discrete parameter $T = n\ell$. First, let us compare $\mathscr{C}^{\mathbb{Z}/\ell}_{n\ell}(U \times S^1, \mathbb{K})$ and $C^{\mathbb{Z}/\ell}_{n\ell}(U, \mathbb{K})$ if $U \subset T^*X$ is symplectic admissible. By Proposition 4.2, the prequantized open set $U \times S^1$ is contact admissible.

Proposition 4.4. *For a symplectic admissible open set $U \subset T^*X$, for $\ell \in \mathbb{N}$, $n \in \mathbb{N}_0$, we have*

$$C^{\mathbb{Z}/\ell}_{n\ell}(U, \mathbb{K}) \cong \mathscr{C}^{\mathbb{Z}/\ell}_{n\ell}(U \times S^1, \mathbb{K}).$$

Proof. Since $\mathscr{P}_{U\times S^1} \cong m^{-1}P_U$, we have

$$\mathscr{P}^{\overset{L}{\boxtimes}\ell}_{U\times S^1} \cong (m^\ell)^{-1} P^{\overset{L}{\boxtimes}}_U,$$

where $m^\ell(\underline{\mathbf{q}}, t^1_1, t^2_1, \ldots, t^1_\ell, t^2_\ell) = (\underline{\mathbf{q}}, t^2_1 - t^1_1, \ldots, t^2_\ell - t^1_\ell)$. Then we have

$$\mathscr{C}^{\mathbb{Z}/\ell}_{n\ell}(U \times S^1, \mathbb{K}) \cong \mathrm{RHom}_{\mathbb{Z}/\ell}\left(P^{\overset{L}{\boxtimes}\ell}_U, m^\ell_* \beta^c_{\ell,n,X}\mathbb{K}[-d]\right).$$

So, we only need to verify that

$$m^\ell_* \beta^c_{\ell,n,X}\mathbb{K} \cong \beta_{\ell,n\ell,X}\mathbb{K}.$$

By proper base change, we only need to assume $X = \mathrm{pt}$ and then show that $m^\ell_* \widetilde{m}^! i^\ell_{n*}\mathbb{K}[-1] \cong s^{\ell!}_t i_{n\ell*}\mathbb{K}$. Direct computation shows that both sides are isomorphic to $\mathbb{K}_{\{(t_1,\cdots,t_\ell): t_1+\cdots+t_\ell = n\ell\}}[\ell - 1]$. $\square$

On the other hand, the constraint $T/\ell \in \mathbb{N}_0$ is adapt to the problem of invariance. As the lifting of a contact isotopy is merely $\mathbb{Z}$-equivariant, the sheaf quantization will only be $\mathbb{Z}$-equivariant (see Remark 1.19). So our discussion on invariance for symplectic version does not applies directly. However a slight modification for the proof of the symplectic invariance works.

Theorem 4.5 ([Chi17, Theorem 4.7]). *Let $\mathscr{U}, \mathscr{U}_1, \mathscr{U}_2$ be contact admissible open sets and let $\mathscr{U}_1 \xhookrightarrow{i} \mathscr{U}_2$ be an inclusion. Then one has, for $\ell \in \mathbb{N}$, $n \in \mathbb{N}_0$,*

1) There is a morphism $\mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(\mathscr{U}_2, \mathbb{K}) \xrightarrow{i^*} \mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(\mathscr{U}_1, \mathbb{K})$, which is natural with respect to inclusions of admissible open sets.

2) For a compactly supported contact isotopy $\varphi : I \times T^*X \times S^1 \to T^*X \times S^1$,. We have an isomorphism, in the equivariant category, $\Phi_{z,n\ell}^{\mathbb{Z}/\ell,c} : \mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(\mathscr{U}, \mathbb{K}) \xrightarrow{\cong} \mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(\varphi_z(\mathscr{U}), \mathbb{K})$, for all $z \in I$. The isomorphism $\Phi_{z,n\ell}^{\mathbb{Z}/\ell,c}$ is functorial with respect to restriction morphisms in (1). When $\mathscr{U} = T^*X \times S^1$, we have $\Phi_{z,n\ell}^{\mathbb{Z}/\ell,c} = \mathrm{Id}$.

The proof for (1) is the same as the symplectic case. Let us present the proof for invariance, which is slightly different from the symplectic one.

Proof of Theorem 4.5 (2). For the contact isotopy φ, we take the GKS quantization $K(\widehat{\varphi'})$ as we discussed in subsection 1.3. Let $K = K(\widehat{\varphi'})_z$, $K_\ell = K^{\overset{L}{\boxtimes}\ell}$ and $K_\ell^{-1} = (K^{-1})^{\overset{L}{\boxtimes}\ell}$.

Recall the proof of Theorem 2.15 (2). In the contact case, we still have an isomorphism

$$\mathscr{P}_{\varphi_z(\mathscr{U})} \cong K^{-1} \circ \mathscr{P}_{\mathscr{U}} \circ K,$$

as well as the auto-equivalence $\kappa(F) := K_\ell^{-1} \circ F \circ K_\ell$ of $D_{\mathbb{Z}/\ell}((X \times \mathbb{R})^{2\ell})$.

So, we only need to construct an isomorphism

$$\kappa(\beta_n^c \mathbb{K}) = K_\ell^{-1} \circ \beta_n^c \mathbb{K} \circ K_\ell \cong \beta_n^c \mathbb{K}, \tag{4.2}$$

where $\beta_n^c = \beta_{\ell,n,X}^c$. As in Theorem 2.15, we only need to find an isomorphism

$$\beta_n^c \mathbb{K} \circ K_\ell \cong K_\ell \circ \beta_n^c \mathbb{K}.$$

To emphasize the difference between the contact case and the symplectic case, let us present the construction precisely. Let $W = X \times \mathbb{R}$, $f \colon W^\ell \to W^\ell$, $(w_1, \dots, w_\ell) \mapsto (w_2, \dots, w_\ell, w_1)$ where $w_i = (\mathbf{q}_i, t_i)$ and $\mathrm{T}_c^\ell : W^\ell \to W^\ell$, $(w_1, \dots, w_\ell) \mapsto (\mathrm{T}_c(w_1), \dots, \mathrm{T}_c(w_\ell))$, where $c \in \mathbb{R}$ and

$\mathrm{T}_c(w_i) = \mathrm{T}_c(\mathbf{q}_i, t_i) = (\mathbf{q}_i, t_i + c)$. Set $Y = W^\ell$ and identify $Y^2 = (W^2)^\ell$ by $(w_1^1, \dots, w_\ell^1, w_1^2, \dots, w_\ell^2) \mapsto (w_1^1, w_1^2, \dots, w_\ell^1, w_\ell^2)$. Then $\beta_n^c \mathbb{K}$ is, up to orientation and shift, the constant sheaf on the graph of the composition $f \circ \mathrm{T}_n^\ell = \mathrm{T}_n^\ell \circ f$. Precisely, we have

$$\beta_n^c \mathbb{K} \cong \mathbb{K}_{\Gamma_f} \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ E \cong E \circ \mathbb{K}_{\Gamma_f} \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}},$$

where $E = \delta_{Y^2 !}(\omega_Y)$, with ω_Y the dualizing sheaf and δ_{Y^2} the usual diagonal embedding. The relation $f \circ \mathrm{T}_n^\ell = \mathrm{T}_n^\ell \circ f$ implies $\mathbb{K}_{\Gamma_f} \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \cong \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ \mathbb{K}_{\Gamma_f}$. Moreover we have $E \circ - \cong - \circ E$.

Now we have the general fact $G \circ \mathbb{K}_{\Gamma_g} \cong (\mathrm{Id}_Y \times g)_!(G)$ for any G and any map g. This formula has the symmetric form $\mathbb{K}_{\Gamma'_g} \circ G \cong (g \times \mathrm{Id}_Y)_!(G)$ where Γ'_g is the switched graph $\Gamma'_g = \{(g(y), y) : y \in Y\}$. When g is invertible, we have $\Gamma_{g^{-1}} = \Gamma'_g$. So we obtain

$$K_\ell \circ \beta_n^c \mathbb{K} \cong K_\ell \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ \mathbb{K}_{\Gamma_f} \circ E \cong (\mathrm{Id}_Y \times f)_!(K_\ell \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}}) \circ E,$$

and

$$\begin{aligned} \beta_n^c \mathbb{K} \circ K_\ell &\cong E \circ \mathbb{K}_{\Gamma_f} \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ K_\ell = E \circ \mathbb{K}_{\Gamma'_{f^{-1}}} \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ K_\ell \\ &\cong E \circ (f^{-1} \times \mathrm{Id}_Y)_!(\mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ K_\ell). \end{aligned}$$

Now, recall the GKS quantization $K(\widehat{\varphi'})$ satisfies the $\mathbb{Z}$-equivariant condition (1.12), so the restriction on z-slices, $K = K(\widehat{\varphi'})_z$, also satisfies

$$K \circ \mathbb{K}_{\Delta_{X^2} \times \{(t, t+n) : t \in \mathbb{R}\}} \cong \mathbb{K}_{\Delta_{X^2} \times \{(t, t+n) : t \in \mathbb{R}\}} \circ K.$$

Notice that $\Delta_{X^2} \times \{(t, t+n) : t \in \mathbb{R}\} = \Gamma_{\mathrm{T}_n}$ where $\mathrm{T}_n(x, t) = (x, t+n)$. Therefore we have

$$K_\ell \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \cong \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ K_\ell.$$

Hence we have

$$K_\ell \circ \beta_n^c \mathbb{K} \cong (\mathrm{Id}_Y \times f)_!(K_\ell \circ \mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}}) \circ E \cong E \circ (\mathrm{Id}_Y \times f)_!(\mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ K_\ell).$$

Then the isomorphism (4.2) follows from

$$\begin{aligned} (\mathrm{Id}_Y \times f)_!(\mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ K_\ell) &\cong (f \times f)_!(f^{-1} \times \mathrm{Id}_Y)_!(\mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ K_\ell) \\ &\cong (f^{-1} \times \mathrm{Id}_Y)_!(\mathbb{K}_{\Gamma_{\mathrm{T}_n^\ell}} \circ K_\ell). \end{aligned}$$

The isomorphism $\Phi_{z,n\ell}^{\mathbb{Z}/\ell,c}$ follows. □

Remark 4.6. The significant difference between $C_T^{\mathbb{Z}/\ell}$ and $\mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}$ is that the definition of α^c also twist t variables while α only twist $\mathbf{q}$ variables. This is crucial for the contact invariance in Theorem 4.5. However Proposition 4.4 shows that when we consider the admissible sets of the form $U \times S^1$ for $U \subset T^*X$, the Chiu-Tamarkin complex itself is not affected by the difference. This is helpful for our computations.

Now, we assume $\ell \in \mathbb{N}_{\geq 2}$ and $\mathscr{U} \subset T^*X \times S^1$ is admissible. Then $H^*\mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(\mathscr{U}, \mathbb{K})$ is a module of $A = \mathrm{Ext}^*_{\mathbb{Z}/\ell}(\mathbb{K}, \mathbb{K})$. For an orientable manifold X and a field $\mathbb{K}$, the *fundamental class* $\eta_{n\ell}^{\mathbb{Z}/\ell,c}(\mathscr{U})$ is defined as the image of the fundamental class $[X]^{\mathbb{Z}/\ell} = [X] \otimes 1$ under the morphism $H_d^{BM}(X, \mathbb{K}) \otimes \mathrm{Ext}^0_{\mathbb{Z}/\ell}(\mathbb{K}, \mathbb{K}) \cong H^0\mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(T^*X \times S^1, \mathbb{K}) \xrightarrow{i^*_{\mathscr{U}}} H^0\mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(\mathscr{U}, \mathbb{K})$. Similarly to Proposition 2.23, Theorem 4.5 shows that the fundamental class is preserved under inclusion and contact isotopy.

For the definition of capacities, it is reasonable to require a discrete spectrum. Let $\mathbb{P}$ denote the set of all prime numbers.

Definition 4.7. For an admissible open set $\mathscr{U} \subset T^*X \times S^1$, $k \in \mathbb{N}$. Define

$$[\mathrm{Spec}](\mathscr{U}, k) := \{p \in \mathbb{P} : \eta_p^{\mathbb{Z}/p,c}(\mathscr{U}, \mathbb{F}_p) \in u^k H^*\mathscr{C}_p^{\mathbb{Z}/p}(\mathscr{U}, \mathbb{F}_p)\}$$

and

$$[c]_k(\mathscr{U}) := \min[\mathrm{Spec}](\mathscr{U}, k) \in \mathbb{P}.$$

For a general open set O, we define

$$[c]_k(\mathscr{U}) = \sup\{[c]_k(O) : O \subset \mathscr{U},\ O \text{ is admissible}\}.$$

Let us discuss the properties of $[c]_k$. The invariance and monotonicity are true with the same proof as in the symplectic case. The proof of representing property is invalid now. The positivity for open sets is obviously true by definition. However it is possible that $[c]_k$ is always 2, which is treated as the trivial situation here. To avoid this situation, we must address some restrictions on the size of domains. Consider the constrain given by the structure theorem, we assume ℓ be a prime number; and moreover, the computation of ball indicate we should take $a > 1$ as a necessary size constraint for $B_a \times S^1$. This fits into the framework of [EKP06] that a small contact ball can be squeezed into smaller contact balls. Therefore, we define

Definition 4.8. For an open set $\mathscr{U} \subset T^*\mathbb{R}^d \times S^1$, we say it is *big* if there is a prequantized ball $B_a \times S^1 \xhookrightarrow{contact} \mathscr{U}$ such that $a > 1$.

In summary, we organize our discussions as the following theorem. In the contact case, the spectrum sets could provide us more interesting obstructions. So we state results of spectrum sets as well.

Theorem 4.9. *The functions* $[c]_k : Open(T^*X \times S^1) \to \mathbb{P}$ *satisfy the following:*

1) $[c]_k \leq [c]_{k+1}$ *and* $[\mathrm{Spec}](\mathscr{U}, k+1) \subset [\mathrm{Spec}](\mathscr{U}, k)$*, for all* $k \in \mathbb{N}$*.*

2) For two open sets $\mathscr{U}_1 \subset \mathscr{U}_2$*, then* $[c]_k(\mathscr{U}_1) \leq [c]_k(\mathscr{U}_2)$ *and* $[\mathrm{Spec}](\mathscr{U}_2, k) \subset [\mathrm{Spec}](\mathscr{U}_1, k)$*.*

3) For a compactly supported contact isotopy $\varphi : I \times T^*X \times S^1 \to T^*X \times S^1$*, we have* $[c]_k(\mathscr{U}) = [c]_k(\varphi_z(\mathscr{U}))$ *and* $[\mathrm{Spec}](\mathscr{U}, k) = [\mathrm{Spec}](\varphi_z^H(\mathscr{U}), k)$*.*

4) When $X = \mathbb{R}^d$ *and* $\mathscr{U} \subset T^*\mathbb{R}^d \times S^1$ *is big, then it cannot happen that* $[c]_k(\mathscr{U}) = 2$ *for all* $k \in \mathbb{N}$*.*

Finally, let us discuss the prequantized toric domains, i.e., $X_\Omega \times S^1$ for a symplectic toric domain X_Ω. We say that $X_\Omega \times S^1$ is convex if X_Ω is convex.

Actually, we do not need to change the arguments much here because we already set everything up well. Using Proposition 4.4, we only need to slightly change the statement of the structural theorem.

Theorem 4.10. *Let* $X_\Omega \times S^1 \subset T^*V \times S^1$ *be a big prequantized convex toric domain (that means* $\|\Omega_1^\circ\|_\infty < 1$*, see Definition 4.8) and* $\ell \in \mathbb{N}_{\geq 2}$*. If* $n \in \mathbb{N}_0$ *and* $n\ell \leq p_\ell / \|\Omega_1^\circ\|_\infty$*, we have:*

- *For each* $Z \in \Omega_{n\ell}^\circ$*, the inclusion* $\overline{ZO} \subset \Omega_{n\ell}^\circ$ *induces a decomposition* $\eta_{n\ell}^{\mathbb{Z}/\ell, c}(X_\Omega \times S^1, \mathbb{F}_{p_\ell}) = u^{I(Z)} \Lambda_{Z,\ell}$ *for a non-torsion element* $\Lambda_{Z,\ell} \in H^{-2I(Z)} \mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$*. In particular,* $\eta_{n\ell}^{\mathbb{Z}/\ell, c}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$ *is non-zero.*
- *The minimal cohomology degree of* $H^* \mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$ *is exactly* $-2I(\Omega_{n\ell}^\circ)$*, i.e.,*

$$H^* \mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell}) \cong H^{\geq -2I(\Omega_{n\ell}^\circ)} \mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$$

and

$$H^{-2I(\Omega_{n\ell}^\circ)} \mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell}) \neq 0.$$

- $H^* \mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$ *is a finitely generated* $\mathbb{F}_{p_\ell}[u]$*-module. The free part is isomorphic to* $A = \mathbb{F}_{p_\ell}[u, \theta]$*, so* $H^* \mathscr{C}_{n\ell}^{\mathbb{Z}/\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$ *is of rank* 2 *over* $\mathbb{F}_{p_\ell}[u]$*.*

The torsion part is located in cohomology degree $[-2I(\Omega^\circ_{n\ell}), -1]$. $H^*\mathscr{C}^{\mathbb{Z}/\ell}_{n\ell}(X_\Omega \times S^1, \mathbb{F}_{p_\ell})$ *is torsion free when* X_Ω *is an open ellipsoid.*

Theorem 4.11. *For a big prequantized convex toric domain* $X_\Omega \times S^1 \subsetneq T^*V \times S^1$*, we have:*

$$[c]_k(X_\Omega \times S^1) = \min\left\{p \in \mathbb{P} : \exists z \in \Omega^\circ_p, I(z) \geq k\right\} = \min\left\{p \in \mathbb{P} : p \geq c_k(X_\Omega)\right\}.$$

Proof. If $p \in [\mathrm{Spec}](X_\Omega \times S^1, k)$, then $p < \frac{p}{\|\Omega^\circ_1\|_\infty}$ and we can use the structure theorem by the bigness condition. Then we use the minimal degree result of Theorem 4.10 to show that $I(z) \geq k$ for some $z \in \Omega^\circ_p$. In particular, we have $p \geq c_k(X_\Omega)$.

Conversely, if prime number p satisfies the condition $p \geq c_k(X_\Omega)$. We can find a $z \in \Omega^\circ_p$, and the existence of decomposition of the fundamental class in Theorem 4.10 implies that $\eta_p^{\mathbb{Z}/p,c}(X_\Omega \times S^1, \mathbb{F}_p) \in u^k H^*\mathscr{C}^{\mathbb{Z}/p}_p(X_\Omega \times S^1, \mathbb{F}_p)$. □

The result is much weaker than the symplectic case, while it is still interesting. For example, when we consider ellipsoids, we have

$$[c]_k(E(a) \times S^1) = \min\left\{p \in \mathbb{P} : \sum_{i=1}^{d} \left\lfloor \frac{p}{a_i} \right\rfloor \geq k\right\} = \min\left\{p \in \mathbb{P} : p \geq c_k(E(a))\right\},$$

where $a = (a_1, \dots, a_d)$ and $1 < a_1 \leq a_2 \leq \cdots \leq a_d$.

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