- GF is a low-tech tool to prove rigidity results in sym. / cont. topology: Arnold conj. / Gromov non-squeezing thm (Viterbo).

- Microlocal theory of sheaves
  Nice interaction with GF.

M smooth mfd. T^*M cotangent bundle with the canonical 1-form λ_M.

Locally, $\lambda_M = \sum_{i=1}^{n} p_i dq_i \Rightarrow d\lambda_M$ symp.

$\sigma: M \to T^*M$ 1-form is Lagrangian iff σ is closed.

σ is exact Lag $\Leftrightarrow$ σ is exact.

$J^1M = T^*M \times \mathbb{R}_z = J^1(M, \mathbb{R})$, $\alpha_M = dz - \lambda_M$ contact form

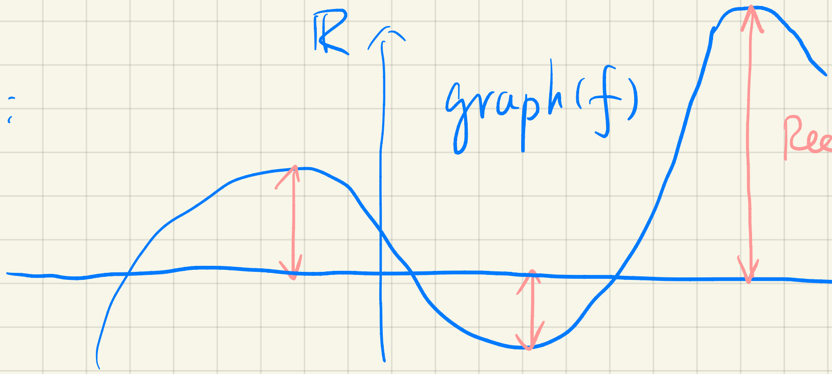
the Reeb field of α_M is $\frac{\partial}{\partial z}$.

$\sigma: M \to J^1M$ is Legendrian iff $\sigma = j^1 f$, for $f: M \to \mathbb{R}$.

$j^1 f(q) = (q, \frac{\partial f}{\partial q}, f)$.

Front projection: $J^1M \to J^0M = M \times \mathbb{R}$.

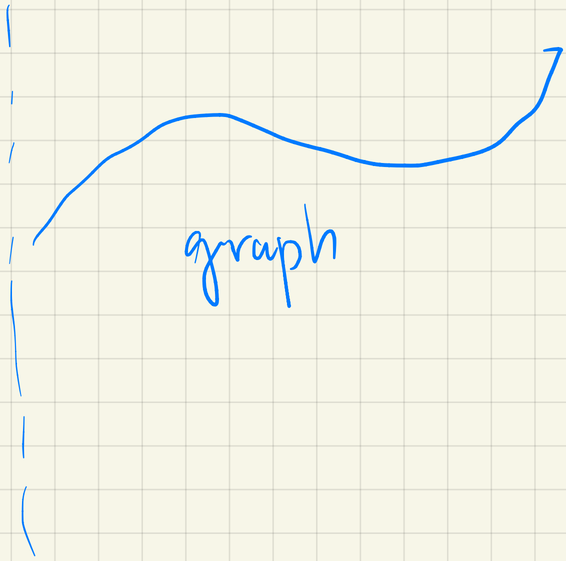
Front proj. of j^1f:



So critical pts of f $\overset{1:1}{\longleftrightarrow}$ $gr(df) \cap O_M$ (the zero section

$\overset{1:1}{\longleftrightarrow}$ Reeb chords of j^1f and O_M

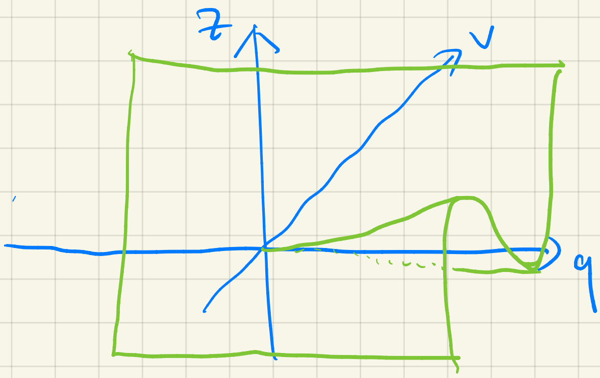
Front proj. of Leg.



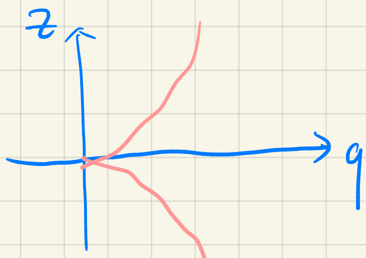
But reduction of a graph with more variable.

Ex: $f: \mathbb{R}_q \times \mathbb{R} \to \mathbb{R}$, $f(q,v) = v^3 - 3qv$.

$M = \mathbb{R}_q$



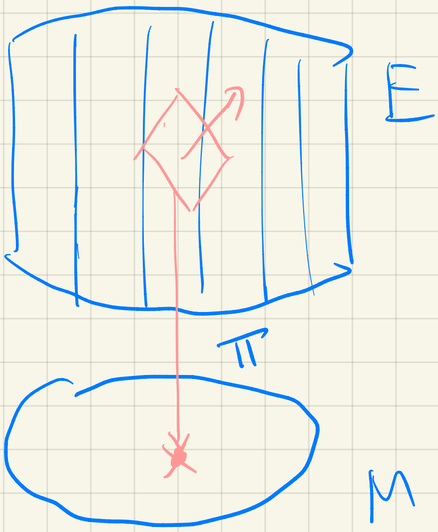
$\xrightarrow[\text{proj.}]{\text{Front}}$



Def A Generating function (gf) over M is a pair of $\begin{cases} \text{submersion} & \pi : E \to M \\ \text{function} & f : E \to \mathbb{R}. \end{cases}$

subject to a (generic) tranversality condition.

$$\left(\begin{tikzcd}
T^*M & T^*M \times_M E \arrow[l] \arrow[r, hook, "\iota"] & T^*E \\
J^1M \arrow[u] & \Sigma_f \arrow[l] \arrow[ul] \arrow[u] \arrow[r, hook] \arrow[ur, phantom, "\square"] & E \arrow[u, "df"']
\end{tikzcd}\right)$$



s.t. $\iota \pitchfork df$

or $df \pitchfork T^*M \times_M E$

$\qquad\qquad = \{ \beta \in T^*E : \beta|_{f \circ d\pi} = 0 \}$

Locally, $E = \mathbb{R}^n \times \mathbb{R}^k$, $\pi(q,v) = q$, $f(q,v)$

$T^*M \times_M E = \{(q, v; p, 0)\} \subseteq T^*E$ fiberwise zero-section

coisotropic.

$\Sigma_f = \{(q,v) : \frac{\partial f}{\partial v} = 0\} \xrightarrow{i_f} J^1M$, $(q,v) \mapsto (q, \frac{\partial f}{\partial q}, f)$.

Claim: under transversality, Σ_f is a submanifold, i_f is a Legendrian immersion.

Linear algebra: $H \subseteq V$ coisotropic subspace.

L Lagrangian subspace, $L \pitchfork H$.

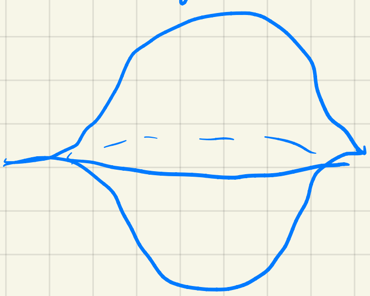
Then $L \cap H \to H/H^{\perp}$ inj with Lag. image.

Ex 1) $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $f(q,v) = v^3 + 3(\|q\|^2 - 1)v$

$\Sigma_f = \{ v^2 + \|q\|^2 = 1 \} \cong S^n$

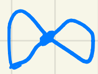
$i_f(q,v) = (q, 6qv, v^3 + 3(\|q\|^2 - 1)v)$

Front: flying saucer



$\Sigma_f \to J^0\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}$

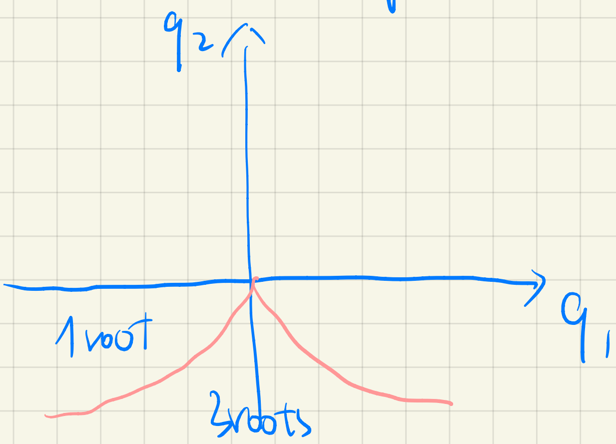
$S^n \hookrightarrow \mathbb{R}^{2n}$ Lag. immersion with a single double pt.

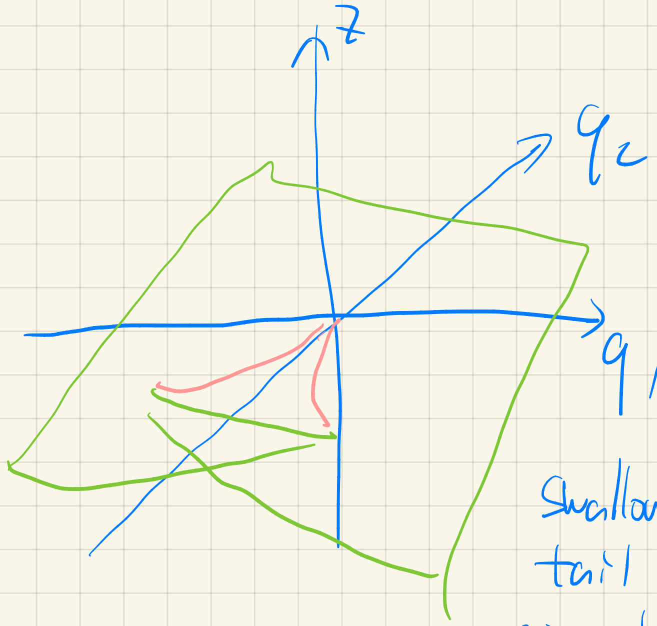
$n=1$: 

2) $f: \mathbb{R}^2_q \times \mathbb{R} \to \mathbb{R}$, $f(q_1, q_2, v) = v^4 + q_2 v^2 + q_1 v$

$\frac{\partial f}{\partial v} = 4v^3 + 2q_2 v + q_1$, has double root along

$27 q_1^2 + 8 q_2^3 = 0$.





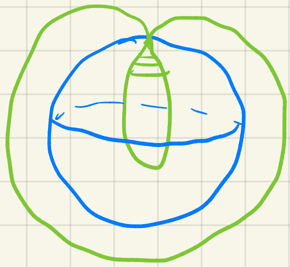
Swallow tail singularities.

3) $S^3 \xrightarrow{f} \mathbb{R}$, $\pi(z_1, z_2) = [z_1, z_2]$.

$\pi \downarrow$ Hopf fibration $f(z_1, z_2) = \operatorname{Im} z_2$.

$S^2 = \mathbb{C}P^1$

$S^2 \times \mathbb{R} \simeq \mathbb{R}^3 \setminus \{0\}$,



Alvarez-Gavela, Igusa for more gf on S^1 bundle

NB: $\pitchfork$ condition means $\frac{\partial f}{\partial v} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ is a submersion.

$\Leftrightarrow$ $\left(\frac{\partial^2 f}{\partial q \partial v}, \frac{\partial^2 f}{\partial v^2} \right)$ has rank $= k$.

critical pts of f: $M \xleftarrow{\pi} E \xrightarrow{f} \mathbb{R}$

$$\{ (q, v) : \frac{\partial f}{\partial q} = \frac{\partial f}{\partial v} = 0 \}$$

$$\|$$

$$\{ (q, v) \in \Sigma_f : (q, \frac{\partial f}{\partial q}) \in O_M \}$$

$\xleftrightarrow{1:1}$ $L_f \cap O_M$ in T^*M $\xleftrightarrow{1:1}$ Reeb chords from L_f to O_M in J^1M.

Morse theory should give lower bounds on

$\# L_f \cap O_M$.

<u>Arnold conj</u> M closed, $\varphi_t : T^*M \to T^*M$

Ham isotopy,

$\Rightarrow |\varphi_1(O_M) \pitchfork O_M| \geq \sum b_i(M)$.

Sketch of the pf: (Laudenbach-Sikorav)

- O_M has a gf:

$$\begin{array}{ccc} M & \xrightarrow{0} & \mathbb{R} \\ \downarrow & & \\ M & & \end{array}$$

- This persists under Ham. isotopy, so $\varphi_1(O_M)$ admits a gf which can taken quadradic at infinity.

in the sense $\begin{array}{c} M_q \times \mathbb{R}^k_v \to \mathbb{R} \\ \downarrow \\ M \end{array}$, $f(q,v) = g(v) + \varepsilon(q,v)$

$g(v)$ is a nondeg. quadratic form.

$\varepsilon(q,v)$ cptly supported.

$\rightsquigarrow H^\bullet(M \times \mathbb{R}^k ; f \leq -\infty) \simeq H^\bullet(M) \otimes H^\bullet(\mathbb{R}^k, g \leq -\infty)$

$\simeq H^\bullet(M)[d]$

cohomology of complex generated by $\mathrm{crit}(f) \simeq \varphi_1(O_M) \cap O_M$.

If $\begin{array}{ccc} E_1 & \xrightarrow{f_1} & \mathbb{R} \\ \downarrow \pi_1 & & \\ M & & \end{array}$, $\begin{array}{ccc} E_2 & \xrightarrow{f_2} & \mathbb{R} \\ \downarrow \pi_2 & & \\ M & & \end{array}$

$E_1 \times_M E_2 \xrightarrow{\delta = f_1 \ominus f_2} \mathbb{R}$, "difference function"

$\mathrm{crit}(\delta) \longleftrightarrow$ Reeb chords from L_{f_2} to L_{f_1}.

Locally, $\delta(q, v_1, v_2) = f(q, v_1) - f(q, v_2)$.

$$\mathrm{crit}(\delta) = \left\{ \frac{\partial f_1}{\partial v_1} = 0, \ \frac{\partial f_2}{\partial v_2} = 0, \ \frac{\partial f_1}{\partial q} = \frac{\partial f_2}{\partial q} \right\}$$

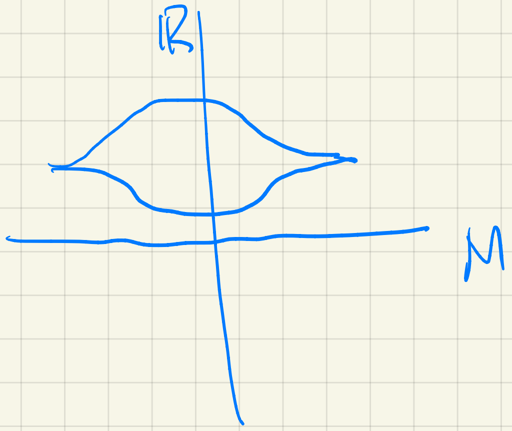
$$= \Sigma_{f_1} \underset{T^*M}{\times} \Sigma_{f_2}$$

If $L_{f_1} = L_{f_2} = L$, δ has a Morse-Bott

submfd in $\mathrm{crit}(\delta)$, a copy of L.

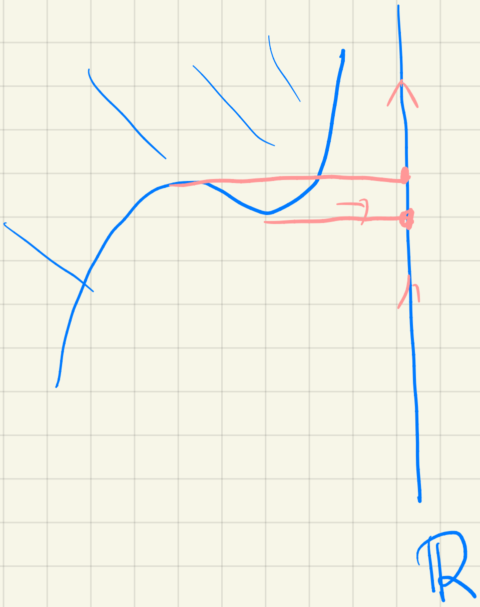
Sheaf associated to g.f.

For a g.f. $\quad E \xrightarrow{(\pi, f)} M\times\mathbb{R}$.



, even better

$$\begin{array}{c} E\times\mathbb{R} \supseteq Z_f = \{(e,z) : z \geq f(e)\} \\ \downarrow \pi\times id \\ M\times\mathbb{R} \end{array}$$



fibers above (q,z) is the sublevel set $\{f_q \leq z\}$

$(M = pt)$ $\quad f_q : \pi^{-1}(q) \to \mathbb{R}$

We associate a sheaf on $M\times\mathbb{R}$, whose stalks at (q,z) is $H^\bullet(\{f_q \leq z\})$

<u>Def</u> A <u>sheaf</u> on X (k ring $\mathbb{Z}$, or $\mathbb{Z}/2$)

- U open set, $F(U)$ k-module.

- $W \subseteq V \subseteq U$,

$$\begin{tikzcd} F(U) \arrow[r] \arrow[dr] & F(V) \arrow[d] \\ & F(W) \end{tikzcd}$$

- For $U = \bigcup_i U_i$,

$$0 \to F(U) \to \bigoplus_i F(U_i) \longrightarrow \bigoplus_{i,j} F(U_i \cap U_j)$$
$$s \mapsto s|_{U_i} \mapsto s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}$$

is exact.

- For $x \in X$, stalk at x is

$$F_x = \varinjlim_{x \in U} F(U).$$

<u>Ex</u> • k_X constant sheaf on X,

$k_X(U) = \{f : U \to k \text{ locally constant}\}$.

- $Z \overset{\text{closed}}{\subseteq} X$, $k_Z(U) = \{U \cap Z \to k \text{ locally constant}\}$.

Morphism of sheaves:

$F \xrightarrow{\varphi} G$,

$\forall U, V$, $V \subseteq U$

$$\begin{tikzcd} F(U) \arrow[r, "\varphi_U"] \arrow[d] & G(U) \arrow[d] \\ F(V) \arrow[r, "\varphi_V"] & G(V) \end{tikzcd}$$

$Mod(k_X)$ = Abelian category of sheaves of k-module on X. $f: Y \to X$ continuous

Operation: $\otimes$, $\mathcal{H}om$, f_*, f^{-1}, $f_!$

$F, H \in Mod(k_X)$, $G \in Mod(k_Y)$,

- $f^{-1}F(U) = \lim\limits_{V \supseteq f(U)} F(V)$ + sheafify.
- $f_* G(U) = G(f^{-1}(U))$
- $\mathcal{H}om(F, H)(U) = \mathrm{Hom}_{Mod(k_U)}(F|_U, H|_U)$

$F|_U = i^{-1}F$, $i: U \to X$.

- $F \otimes H(U) = F(U) \otimes H(U)$ + sheafify.
- $f_! G(U) = \{ s \in G(f^{-1}(U)) : f|_{supp(s)} : supp(s) \to X \text{ is proper} \}$

Cohomology of sheaves

$Mod(k_X) \xrightarrow[\text{category}]{\text{derived}} D^b(k_X)$

- bounded complex of sheaves $0 \to \dots \to F^k \to F^{k+1} \to \dots \to 0$

$F^\cdot \to G^\cdot$ is an iso iff $H^i(F^\cdot) \to H^i(G^\cdot)$ iso for all i. (quasi-isomorphism).

Derived functor: $Rf_*: D^b(k_Y) \to D^b(k_X)$

Ex: k_X constant sheaf. $k = \mathbb{R}$, X closed mfd.

$\Gamma(k_X) = k_X(X) = \{X \to \mathbb{R}$ locally constant$\}$

$= H^0(X; \mathbb{R})$

$\rightsquigarrow$ Replace k_X by a quasi-iso. complex which is Γ-acyclic

e.g. de Rham complex.

$$\begin{array}{ccccccccccc} 0 & \to & \Omega_X^0 & \to & \Omega_X^1 & \to & \cdots & \to & \Omega_X^n & \to & 0 \\ & & \uparrow & & & & & & \uparrow & & \\ 0 & \to & k_X & \to & 0 & \to & \cdots & \to & 0 & \to & 0 \end{array}$$

qis by the Poincaré Lemma.

So $R\Gamma(k_X) \simeq \Gamma(\Omega_X^\bullet)$ in $D^b(k_X) \xrightarrow{H^i} \mathrm{Mod}(k_X)$

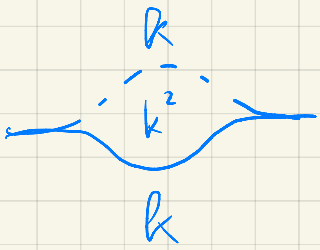
and $H^i(R\Gamma(k_X)) \cong H^i_{dR}(X; \mathbb{R})$

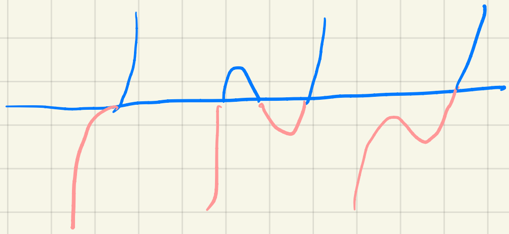
Ex $E \times \mathbb{R} \supseteq Z_f \xrightarrow{\pi \times id} M \times \mathbb{R}$

$F_f := R(\pi \times id)_* k_{Z_f}$ (or variants $\pi_!$)

in good situation, $(F)_{(q,z)} \simeq H^*(f_q \le z)$.

EX





$f(q,v) = v^3 + 3(q^2-1)v$.

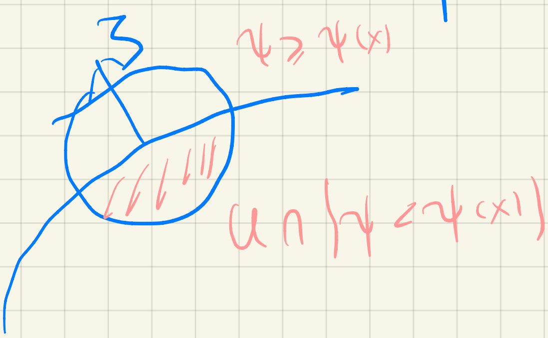
(Micro support, Kashiwara – Schapira)

Def: $F \in D^b(k_X)$, $SS(F) \subseteq T^*X$ is in the closure of the set $(x,\xi) \in T^*X$

s.t. $\exists \psi : X \to \mathbb{R}$, $d_x\psi = \xi$, $\exists i \in \mathbb{Z}$.

s.t. $\varinjlim_{x \in U} H^i(U, F) \longrightarrow \varinjlim_{x \in U} H^i(U \cap \{\psi < \psi(x)\}; F)$

is not an isomorphism.



$SS(F) = \{$ codirection where section of sheaves F do not propogate well $\}$.

For F_f associated to a gf $M \xleftarrow{\pi} E \xrightarrow{f} \mathbb{R}$.

Claim: for nice gf f (e.g quadratic at ∞),

we have

$$\begin{array}{ccc} SS(F_f)\setminus 0_{M\times\mathbb{R}} & \simeq & L_f\times\mathbb{R}_+^* \\ \cap & & \cap \\ T^*(M\times\mathbb{R})\setminus 0_{M\times\mathbb{R}} & \leftarrow & J^1M\times\mathbb{R}_+^* \end{array}$$

N.B. $SS(F)$ is closed and $\mathbb{R}_+^*$-invariant (conic)

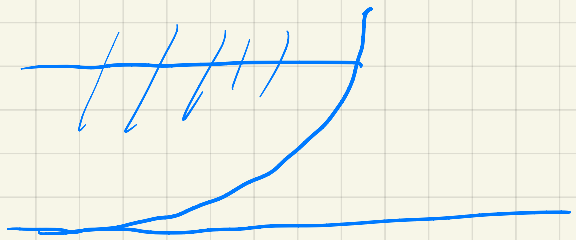
$$\mathbb{R}_+^* = (0,+\infty)$$

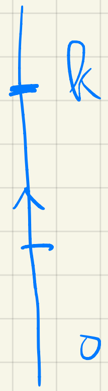
$$T_+^*(M\times\mathbb{R}) = \{(q,z,-sp,s) : s>0\}$$

$$\begin{array}{ccc} T_+^*(M\times\mathbb{R}) & & (q,z,-sp,s) \\ \cong & & \downarrow \\ J^1M\times\mathbb{R}_+^* & & ((q,p,z),s) \end{array}$$

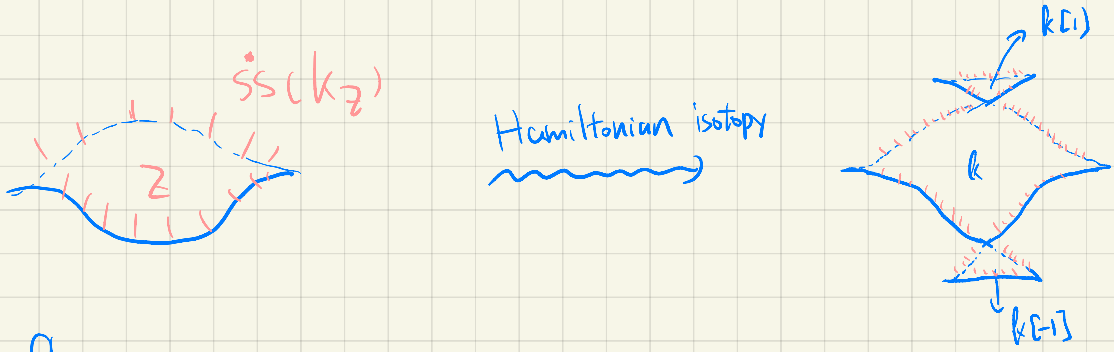
$$s(dz-pdq) = -spdq+sdz.$$

$SS(F)\neq L_f\times\mathbb{R}_+^*$ because change of cohomology of $\{f_q\le z\}$ can hidden at ∞.





Ex



flying saucer

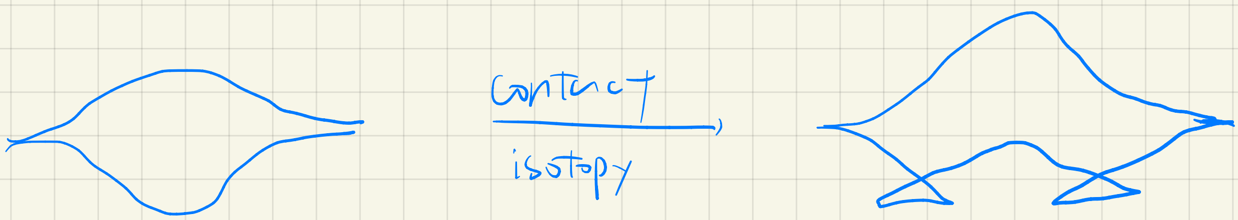
So $SS(R\pi_{Z*}(k_Z))$

$$= \{\infty\} \times \{f(q,v)\} \times \mathbb{R}_+^*$$

in
$T^*\mathbb{R}_q$

$$f(q,v) = v^3 + 3(q^2-1)v$$

Homotopy lifting property for sheaves / GF.



Thm (Guillermou–Kashiwara–Schapira) X mfd,

$\varphi : \dot{T}^*X \times [0,1] \to \dot{T}^*X$ homogeneous Hamitonian

isotopy ($\varphi_t^* \lambda_X = \lambda_X$) ($\Leftarrow$ contact isotopy of ST^*X)

$\dot{T}^*X = T^*X \setminus 0_X$. $\quad \dot{SS}(F) = SS(F) \cap \dot{T}^*X$.

There exists $K\varphi \in D^b(X \times X \times [0,1])$ satisfying :

- $K\varphi|_{t=0} = k_{\Delta_X} \in D^b(X \times X)$, $\Delta_X \subseteq X \times X$ diagonal
- $\dot{SS}(K\varphi) = \Gamma\varphi$

$$= \{ (x, \xi, -\varphi_t(x,\xi), t, \lambda_X(\tfrac{\partial\varphi}{\partial t})) : (x,\xi) \in \dot{T}^*X \}$$

$$\lambda_{X \times X \times [0,1]}\Big|_{\Gamma\varphi} = -\varphi_t^*\lambda - \lambda_X(\tfrac{\partial\varphi}{\partial t})dt + \lambda_X + \lambda_X(\tfrac{\partial\varphi_t}{\partial t})dt = 0.$$

$\rightsquigarrow \Gamma\varphi$ is a conic Lagrangian submanifold.

For $t \in [0,1]$, $K\varphi_t := K\varphi|_t$.

Then $\dot{SS}(K\varphi_t) = \Gamma\varphi_t = \{ (x, \xi, -\varphi_t(x,\xi)) : (x,\xi) \in \dot{T}^*X \}$

$K\varphi_t$ is the integral kernel associated to φ_t.

Composition of kernels

$K_{12} \in D^b(X_1 \times X_2)$, $K_{23} \in D^b(X_2 \times X_3)$

$$\begin{tikzcd}
& X_1 \times X_2 \times X_3 \arrow[dl, "q_{12}"'] \arrow[d, "q_{13}"] \arrow[dr, "q_{23}"] & \\
X_1 \times X_2 & X_1 \times X_3 & X_2 \times X_3
\end{tikzcd}$$

$$K_{12} \circ K_{23} := Rq_{13!}\left(q_{12}^{-1} K_{12} \otimes q_{23}^{-1} K_{23} \right)$$

Under some technical condition,

$$\dot{S}S(K_{12} \circ K_{23}) \subseteq \underbrace{\dot{S}S(K_{12})}_{=\Lambda_{12}} \circ \underbrace{\dot{S}S(K_{23})}_{=\Lambda_{23}}$$

$\Lambda_{12} \circ \Lambda_{23}$

$= \{ (x_1, \xi_1, x_3, -\xi_3) : \exists (x_2, \xi_2)$, s.t.

$(x_1, \xi_1, x_2, -\xi_2) \in \Lambda_{12}$, $(x_2, \xi_2, x_3, -\xi_3) \in \Lambda_{23} \}$

e.g. $\Gamma_{\varphi_{12}} \circ \Gamma_{\varphi_{23}} = \Gamma_{\varphi_{12} \circ \varphi_{23}}$

If $F \in D^b(X)$, $K_{\varphi_t} \in D^b(X \times X)$

Then $F \circ K_{\varphi_t}$ is defined (take $X_1 = pt$, $X_2 = X_3 = X$)

and $\dot{S}S(F \circ K_{\varphi_t}) = \varphi_t(\dot{S}S(F))$

Sketch of the pf of GKS thm:

- Start with k_{Δ_X}. ($F \circ k_{\Delta_X} \cong F$, so k_{Δ_X} is the kernel of the identity map)

Fact: if $Z \subseteq X$ submfd, then $\dot{S}S(k_Z) = \dot{T}^*_Z X$.

conormal bundle of Z in X.

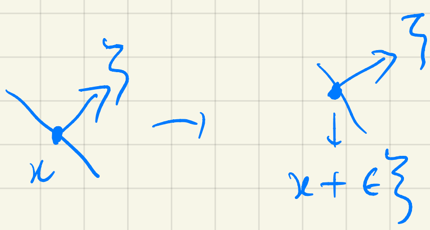
$T^*_{\Delta_X}(X \times X) = \{(x, \xi, x, -\xi)\} = \Gamma_{id}$

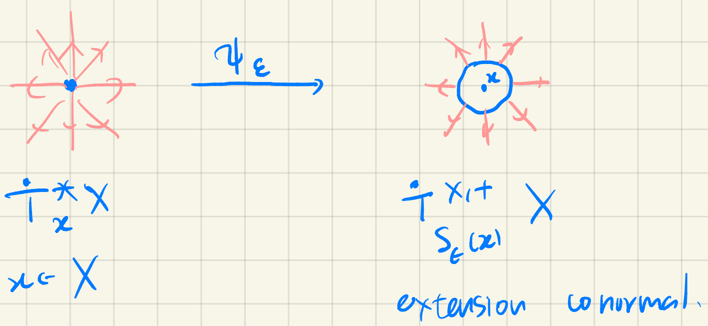
Δ_X is a submfd of codim = dim X.

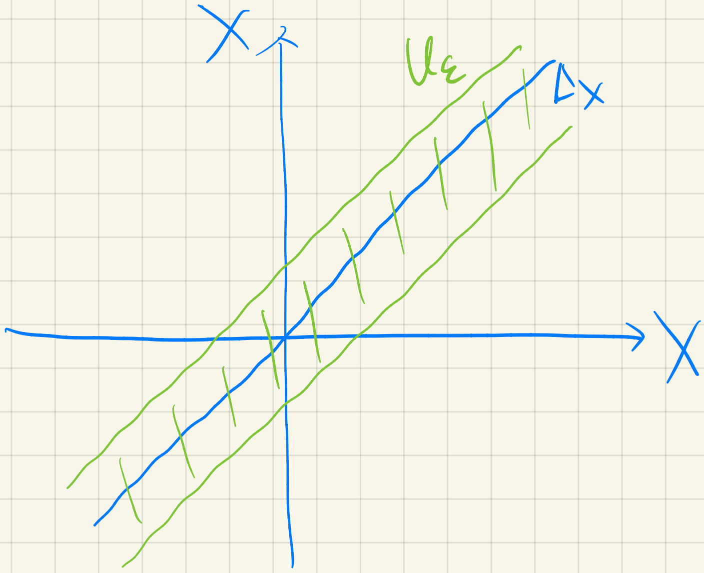
→ not a generic situation

- Pick a Riemannian metric, and $\varepsilon > 0$ small

$\psi_\varepsilon : \dot{T}^* X \to \dot{T}^* X$ normalized geodesic flow at time ε.







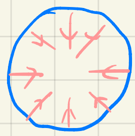
$U_\varepsilon = \{(x,x') \in X \times X : d(x,x') < \varepsilon\}$
open when ε small.

$SS(k_{U_\varepsilon}) = \Gamma_{\psi_\varepsilon} \in D^b(X \times X)$

∂U_ε is a hypersurfaces,
a stable situation.

If $\varphi : \dot{T}^*X \to \dot{T}^*X$ is close to id,
then $\varphi \circ \psi_\varepsilon$ is close to ψ_ε,
and $\Gamma_{\varphi \circ \psi_\varepsilon}$ is the extension conormal of $\tilde{U}_\varepsilon$,
a deformation of U_ε.
So $SS(k_{\tilde{U}_\varepsilon}) = \Gamma_{\varphi \circ \psi_\varepsilon}$.

$(\psi_\varepsilon)^{-1} = \psi_{-\varepsilon}$ satisfies



And $SS(k_{\overline{U}_\varepsilon}) = \Gamma_{\psi_{-\varepsilon}}$, $\overline{U}_\varepsilon = \{ d(x,x') \le \varepsilon \}$

Now, $\varphi = (\varphi \circ \psi_\varepsilon) \circ \psi_\varepsilon^{-1}$ $\rightsquigarrow$ composition of kernels.

For general φ, we fragment it into factors close to id, $\varphi = \varphi_1 \circ \cdots \circ \varphi_N$, φ_i close to id, $\forall i$

$$= [(\varphi_1 \circ \psi_\varepsilon) \circ \psi_\varepsilon^{-1}] \circ [(\varphi_2 \circ \psi_\varepsilon) \circ \psi_\varepsilon^{-1}] \circ \cdots \circ [(\varphi_N \circ \psi_\varepsilon) \circ \psi_\varepsilon^{-1}]$$

Then its kernel is obtained by a sequence of compositions of kernels.

Back to GF: similar story.

$\varphi: T^*M \to T^*M$ exact symplectomorphism ($\varphi^*\lambda_M - \lambda_M$ is exact)

$\Gamma_\varphi = \{(q,p, -\varphi(q,p))\}$ exact Lag. submfd.

<u>Def</u> A gf for φ is a gf for Γ_φ,

i.e.

$$\begin{array}{ccc} E & \xrightarrow{f} & \mathbb{R} \\ \downarrow & & \\ M\times M & & \end{array}$$

generating

Locally, $\pi(q,Q,v) = (q,Q)$

$f(q,Q,v)$

$$\Gamma_\varphi = \left\{\left(q,Q,\frac{\partial f}{\partial q},\frac{\partial f}{\partial Q}\right) : \frac{\partial f}{\partial v}=0\right\}$$

$$= \{(q,Q,p,-P) : \varphi(q,p) = (P,Q)\}$$

Composition

$M_1\times M_2 \leftarrow E_{12} \xrightarrow{f_{12}} \mathbb{R}$, $M_2\times M_3 \leftarrow E_{23} \xrightarrow{f_{23}} \mathbb{R}$

$E_{12}\times_{M_2} E_{23} \xrightarrow{f_{13}} \mathbb{R}$, $\quad f_{13}(e_{12},e_{23}) = f_{12}(e_{12}) + f_{23}(e_{23})$

$\downarrow$

$M_1\times M_3$

$L_{f_{13}} = L_{f_{12}} \circ L_{f_{23}}$

Thm (Sikorav) If $L \to T^*M$ Lagrangian immersion admits a gfqi, then for any compactly supported Ham isotopy $\varphi_t : T^*M \to T^*M$, then $\varphi_1(L)$ also admits a gfqi.

- $\Gamma_{id} = \{(q, p, q, -p)\} \to M \times M$ not surjective.

  Γ_{id} is not a Lag. section of $T^*(M \times M)$

  $\to$ need to deform to a Lag. graph.

  - either use the geodesic flow (Sikorav)
  - or embed $M \hookrightarrow \mathbb{R}^N$ (Chekanov)

Case: $M = \mathbb{R}^n$, $\psi(q, p) = (q+p, p)$ (time 1 map of geodesic flow)

$$\Gamma_\psi = \{(q, p, Q-q, q-Q)\}$$

$$= \{(q, Q, \frac{\partial f}{\partial q}, \frac{\partial f}{\partial Q})\}$$

where $f(q, Q) = -\frac{1}{2}\|q - Q\|^2$

If φ is close to id, so $\Gamma_{\varphi \circ \psi}$ is the graph of $d\tilde{f}$

In general, $\varphi = \varphi_1 \circ \cdots \circ \varphi_N$, φ_i close to id, $\forall i$

$$= [(\varphi_1 \circ \psi) \circ \psi^{-1}] \circ [(\varphi_2 \circ \psi) \circ \psi^{-1}] \circ \cdots \circ [(\varphi_N \circ \psi) \circ \psi^{-1}]$$

ψ^{-1} has gf $-f = \frac{1}{2}\|q-Q\|^2$

$E \xrightarrow{f} \mathbb{R}$ for $L \to T^*M$
$\downarrow$
$\mathbb{R}^n$ $\longrightarrow$ add $2N$ in auxiliary variable when composing with φ.

<u>NB</u> The qi property is not preserved on this process.

So we need to do various cut-offs.

<u>Thm</u> (Chekanov, Chaperon–Théret)

$L \to J^1M$ Legendrian immersion, φ_t contact isotopy of J^1M (cptly supported).

If L has a gfqi, then so does $\varphi_1(L)$

This follows from the symplectic case on $T^*(M\times\mathbb{R})$ by symplectization.

<u>Def</u> $E \xrightarrow{f} \mathbb{R}$, $\mathbb{R}_+^* \circlearrowright E$, $E = E' \times \mathbb{R}_+^*$,
$\downarrow \pi$
$X = M\times\mathbb{R}$ π inv, f equi

$\Rightarrow (\pi, f)$ a homogeneous GF.

Locally, $f(q,v,s) = s f(q,v,1)$, $\pi(q,v,s) = \pi(q,v)$

Σ_f is invariant under $\mathbb{R}_+^*$

$\rightsquigarrow$ generates a conic Lag. in $\dot{T}^*X$.

If
$$\begin{array}{ccc} E & \xrightarrow{f} & \mathbb{R} \\ \downarrow{\scriptstyle \pi} & & \\ M & & \end{array}$$
is a gf.

Then
$$\begin{array}{ccc} E\times\mathbb{R}\times\mathbb{R}_+^* & \xrightarrow{\tilde{f}} & \mathbb{R} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \\ M\times\mathbb{R} & & \end{array}$$
, $\hat{f}(q,z,v,s) = s(z - f(q,v))$

Check: $L_{(\tilde{\pi},\tilde{f})} \cong L_{(\pi,f)} \times \mathbb{R}_+^*$

under iso. $T^{*,+}(M\times\mathbb{R}) \xrightarrow{\cong} J^1M\times\mathbb{R}_+^*$

$(q,z,-ps,s) \longmapsto (q,p,z,s)$

Ex: φ_ε normalized geodesic flow

$\dot{T}^*X \to \dot{T}^*X$

$\dot{SS}(\Bbbk_{\{d(x,x') < \varepsilon\}}) = \Gamma_{\varphi_\varepsilon}$,

$$\begin{array}{ccc} X\times X\times\mathbb{R}_+^* & \xrightarrow{\tilde{f}} & \mathbb{R} \\ \downarrow & & \\ X\times X & & \end{array}$$
, $\tilde{f}(x,x',s) = s(d(x,x') - \varepsilon)$

then $L_{\tilde{f}} = \Gamma_{\varphi_\varepsilon}$

Sheaf $F = R\pi_*(\Bbbk_{\{f<0\}})$, $\dot{SS}(F) = L_f$

Uniqueness or classification of sheaves of Sh / GF for given Legendrian.

If f is a gfqi for $\varphi_1(O_M)$ in T^*M,

$f: M \times \mathbb{R}^k \to \mathbb{R}$

$$\begin{array}{c} H^\bullet(M \times \mathbb{R}^k, \{f \le -\infty\}) \cong H^\bullet(M)[d] \\ \downarrow r_z \\ H^\bullet(\{f \le z\}, \{f \le -\infty\}) \end{array}$$

If $\alpha \in H^\bullet(M)$, let $c(\alpha, f) = \inf\{z : r_z(\alpha) \neq 0\}$.

Thm (Viterbo - Théret) If $L = \varphi_1(O_M)$, then any two gfqi are related by the following operations.

- stablization $M \times \mathbb{R}^k \xrightarrow{f} \mathbb{R} \rightsquigarrow M \times \mathbb{R}^k \times \mathbb{R}^p \xrightarrow{f \oplus Q} \mathbb{R}$
  Q is a fiberwise non-deg. quadratic form
- fiberwise - diffeomorphism:

$$\begin{tikzcd} E \arrow[rr, "\simeq"] \arrow[dr, "(\pi, f)"'] & & E' \arrow[dl, "(\pi', f')"] \\ & M \times \mathbb{R} & \end{tikzcd}$$

So $c(\alpha, f)$ depends only on L_f, not f.

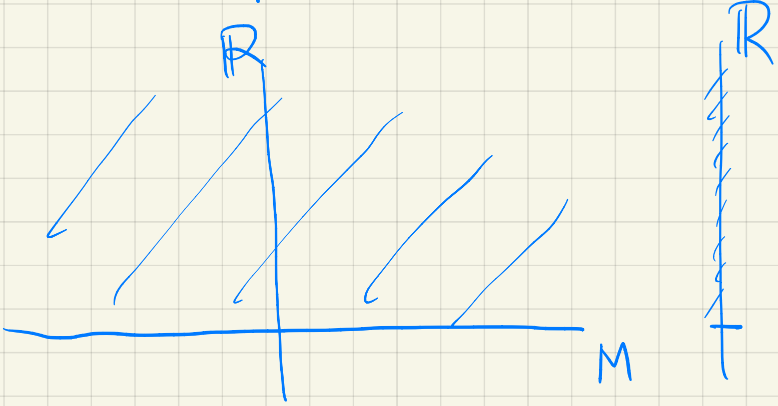
Then Viterbo defined capacities of domains using this $c(\alpha, f)$ (reproved non-squeezing thm)

<u>NB</u> This uniqueness result also holds for sheaves.

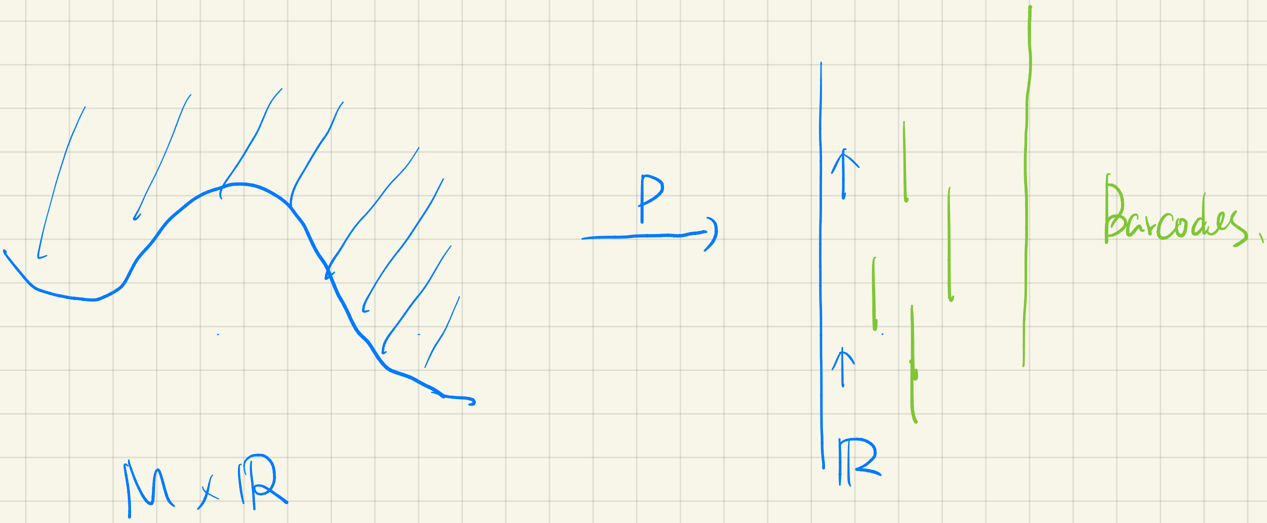
For $F \in D^b(M \times \mathbb{R})$,

if $\dot{SS}(F) = \{(q,0,0,s) : s > 0\}$

then up to local system, $F \cong k_{M \times [0,+\infty)}$



as a corollary of microlocal Morse lemma of sheaves.

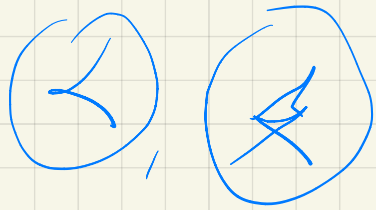


$$D^b(\mathbb{R}) \ni Rp_* F \cong \bigoplus_i k_{[a_i, b_i)}[d_i] \oplus k_{[c_i, +\infty)}[e_i]$$

$\downarrow$

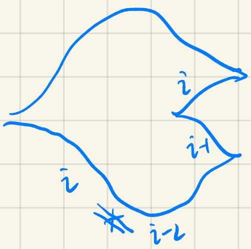
sheaf barcode

What about other Legendrians than $\varphi_1(O_M)$?

- GF / Sh always exist locally.  , ...

What if we try to glue local GF/Sh into global objects?

<u>Ex</u>



NO GF/Sh.

If not, then it assign a morse indices to each branches of the front projection.

$\rightsquigarrow$ first Maslov class obstruction, $\mu_1 \in H^1(L, \mathbb{Z})$:

$L = \bigcup_i L_i$, f_i gf for L_i.

to glue, need isomorphism on $L_{ij} = L_i \cap L_j$

GF: $f_i \ominus f_j$ difference function has L_{ij} as a Morse-Bott submfd in its critical locus, so transversely a non-deg. quadratic form q_{ij},

$\mathrm{sgn}(q_{ij}) \in \mathbb{Z} \rightsquigarrow \mu_1 \in H^1(L, \mathbb{Z})$

Sh: $\mu hom(F_i, F_j) \in D^b(T^*(M\times\mathbb{R}))$, $\dot{SS}(F) \subseteq L_i \times \mathbb{R}_+^*$

$\quad \cong \qquad \longleftarrow$ by microsupport condition

$\quad k_{L_{ij}}[d_{ij}]$, $[\{d_{ij}\}] = \mu_1 \in \check{H}^1(L,\mathbb{Z}) \doteq H^1(L,\mathbb{Z})$

Next,

If $\mu_1 = 0$, $\Rightarrow$

$\varphi_{ij} : F_i \xrightarrow{\sim} F_j$ on L_{ij} with coboundary condition

$\varphi_{ij}\varphi_{jk}\varphi_{ki}$ element in k^*

$\Rightarrow$ class $w_2 \in H^2(L, k^*)$ if $k = \mathbb{Z}$.

w_2 is the Stiefel-Whitney class.

μ_1 and w_2 are only obstruction for sheaves over $\mathbb{Z}$.

Heristically, for gf, the isomorphism φ_{ij}

$\rightsquigarrow E^+(q_{ij}) - E^-(q_{ij})$ a virtual bundle.

$$\begin{array}{c} E^+(q_{ij}) - E^-(q_{ij}) \\ \downarrow \\ L_{ij} \end{array}$$

$\Rightarrow \varphi_{ij}\varphi_{jk}\varphi_{ki} \in O$, stabalized orthogonal group.

$\Rightarrow L \to BO$

Together with μ_1, the obstruction for g.f.

$L \to B(\mathbb{Z} \times BO) \simeq U/O$ Lagrangian Gauss map

$\downarrow$ stablized Lag. Grassmannian.

<u>Thm</u> (Giroux-Latour) $L \to J^1 M$ admits a

gf (in a weak sense, not gfqi) $\Leftrightarrow$

the Lag. Gauss map $L \to U/O$ is nullhomotopic.