Symplectic Non-squeezing Results through Sheaf Theory

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Abstract

In this paper, we review fundamental notions of symplectic geometry and microlocal sheaf theory first. Then we construct a square projector K_∞ in the sense, $\rho(SS(F)) \subset \square$ if and only if $K_\infty \star F \cong F$, and generalize the square projector to cube and prism projector. We estimate their sheaf displacement energy. As applications, we prove some non-squeezing results.

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1 Introduction

Symplectic geometry is a mathematical formulation of the Hamiltonian mechanics at its beginning. In its early life, people thought symplectic geometry was flexible. In fact, because of the Darboux-Weinstein theorem, symplectic manifolds do not admit any local invariants like curvatures in Riemannian geometry. Besides, the difference between volume preserving geometry and symplectic geometry were not well understood as well. The breakthrough happened in [Gro85], Gromov introduced the pseudo-holomorphic curves as a new tool to study symplectic geometry. In particular, he proved that the famous Gromov non-squeezing theorem:

Theorem (Gromov [Gro85]). Equip $\mathbb{R}^{2d}$ with the linear symplectic structure. Let $B_r = \{(x,p) \in \mathbb{R}^{2d} : |x|^2 + |p|^2 < r^2\}$, and $Z_R = \{(x,p) \in \mathbb{R}^{2d} : x_1^2 + p_1^2 < R^2\}$.

If there is a symplectic embedding $\phi : B_r \to \mathbb{R}^{2d}$, such that $\phi(B_r) \subset Z_R$ then $r \leq R$.

It is easy to map the ball B_r into Z_R by volume-preserving squeezing maps when $r > R$. So the Gromov non-squeezing theorem reflects new features of symplectic geometry different from the volume-preserving geometry, which we call the symplectic rigidity phenomenons now. After then, Floer's works on Arnold's conjecture and predictions from the string theory, mirror symmetry in particular, introduce many new theories like Floer cohomology and Fukaya category, embedding contact homology. They can also be used in study of symplectic rigidities, see [MS12] for example. Besides, Viterbo introduced the generating function method into symplectic geometry. In [Vit92], he used generating functions and spectrum invariants to give a new and easy proof of the Gromov non-squeezing theorem.

Almost the same time with Gromov, Kashiwara and Schapira developed the microlocal sheaf theory motivated by researches of ring of (micro-)differential opeartors. Microsupport is used to study propagation of a sheaf. Preciously, for a sheaf $F \in D(M)$, and $(x,p) \in T^*M$, We say F does not propagate at (x,p) if the natural restriction map $\mathcal{H}^i F_x \to \varinjlim_{x \in B} H^i\left(B \cap \{f < f(x)\}, F\right)$ is an isomorphism for all $i \in \mathbb{Z}$ and for all functions f with $df(x) = p$. The microsupport $SS(F)$ is defined by the closure of all propagating codirections of F, which is a conic closed subset of cotangent bundle T^*M. Even though Kashiwara and Schapira did not start from symplectic geometry, they had also noticed the connections between microlocal sheaf theory and symplectic geometry of cotangent bundle, they proved in [KS90] that the microsupport of a sheaf is coisotropic. In particular, they proved that for real analytic manifolds the microsupport of a sheaf is Lagrangian if and only if the sheaf is (weakly-$\mathbb{R}$-)constructible.

Further studys on global aspects of symplectic geometry using the microlocal sheaf theory do not appear in the 80's until work of Bondal-Ruan, Nadler-Zaslow [NZ09, Nad09] and Tamarkin [Tam13]. Bondal claimed a derived equivalence between the Fukaya category of a smooth projective toric variety and a category of constructible sheaves, and Bondal-Ruan announced a proof of homological mirror symmetry of weighted projective space based on the derived equivalence. The paper [NZ09, Nad09] of Nadler-Zaslow also proved a derived equivalence between the infinitesimal Fukaya category of cotangent bundle and tge category of constructible sheaves on the base of the cotangent bundle using microsupports. In [Tam13], Tamarkin established some non-displaceability results in symplectic geometry using properties of microsupports. Moreover, Tamarkin introduced a extra variable t to work on $D(M \times \mathbb{R}_t)$, which could both be used to translate non-conic symplectic geometry problems to the conic world, and to obtain some numerical informations related to symplectic rigidity by the notions of sheaf displacement energy and torsion objects.

In fact, study on non-squeezing problems through the microlocal sheaf theory was seeded in the paper [Tam13]. After then, Chiu realized ideas of Tamarkin, in [Chi17]. Chiu constructed a pair of projectors assoaciate with cantact ball in the contact manifold $\mathbb{R}^{2d} \times S^1$, whose essential image are the category of sheaves whose microsupport contained in the contact ball and its left orthogonal complement. Using the projectors and its homologies in $\mathbb{Z}/N\mathbb{Z}$-equivaraint derived categories, Chiu proved the contact non-squeezing in large radius conjectured in [EKP06].

Our idea is very close to Chiu's. We construct a projector associate with square, which could be generalized to cube and prism, and study their sheaf displacement energy afterwards. We can read the side length of a cube(prism) from the displacement energy of our projector, then we could build some comparison inequality and Hamiltonian invariance to obtain non-squeezing theorem. Our method is a easy and simple model of paper of Chiu, so it is no wonder to believe our method is possible to prove symplectic non-squeezing results.

Finally, let's explain the structure and the main result of the paper. In the following two sections, we review background knowledge about symplectic geometry and microlocal sheaf theory. In particular, we introduces the Tamarkin's cone trick used to translate non-conic symplectic geometry problems into conic world, and introduces sheaf displacement energy and its simple properties as the tools we will use later. In the last section, we constructed a sheaf called square projector, and then estimate its energy upper bound. Then we generalize easily the square projector to case of cubes and prisms, say

Theorem (subsection 4.3)*.* For give size index $\mathbb{s} \in (0, \infty]^{d+1} \times 2^{[d]}$, there is a sheaf $K(\mathbb{s})$ associate to the cube

$$C^d(\mathbb{s}) = \{(x, p) : 2|x_i| \leq s_\alpha, 2|p_\alpha| \leq l_\alpha, \alpha \notin I(\mathbb{s})\},$$

such that for $F \in \mathcal{D}^{t+}(E)$, there is

$$K(\mathbb{s}) \overset{t}{\star} F \to F \text{ is an isomorphism if and only if } \rho(SS(F) \cap T^*_{\tau>0}(M \times \mathbb{R}_t)) \subset C^d(\mathbb{s}).$$

Its displacement energy admits an upper bounded $e(K(\mathbb{s})) \leq min\{ls_\alpha : \alpha \notin I(\mathbb{s})\}$, if $\mathrm{char}(k) \neq 2$.

As application, we present some non-squeezing results in subsection 4.4.

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2 Foundation for Symplectic Geometry

2.1 Basic notions We will review some fundamental notions on symplectic geometry, and then introduce some results we will use later. For more details, one can see [MS17].

Definition 2.1 (Symplectic manifolds)**.** Let X be a smooth manifold, $\omega \in \Omega^2(X)$ be a 2-form on X.

1. We say ω is non-degenerated if ω_x are non-degenerate bilinear functions on T_xX for any $x \in X$. If in addition ω is closed, we call ω a symplectic form on X.

2. If ω is a symplectic form on X, we call (X, ω) a symplectic manifold. If ω is exact, i.e. $\omega = d\alpha$ for some 1-form α, we call $(X, d\alpha)$ a exact symplectic manifold.

Remark 2.2. 1. If (X, ω) is a (exact) symplectic manifold, $(X, -\omega)$ is a (exact) symplectic manifold also. For simplify, we omit ω sometimes, and just say X is a symplectic manifold, then we use $\overline{X}$ to denote $(X, -\omega)$.

2. As ω is non-degenerated, the dimension of X must be even. Besides, as ω non-degenerated if and only if $\omega^n \neq 0$, there is a natural orientation on a symplectic manifold. In particular, exact symplectic manifolds are NOT closed manifolds.

Example 2.3. 1. Let $X = \mathbb{R}^{2n}$, whose coordinate is (x^i, p_i). Let $\alpha = \sum_i p_i dx^i$, then $\omega = d(-\alpha) = \sum_i dx^i \wedge dp_i$ is a exact symplectic form. It provide us a exact symplectic structure on $\mathbb{R}^{2n}$. We call it the linear symplectic structure.

2. Let M be a smooth manifold. T^*M is the cotangent bundle of M. Take a local coordinate (x^i, p_i). $\alpha_M = \sum_i p_i dx^i$ is a well-defined global 1-form, so does $\omega_M = d(-\alpha_M)$. Then they provide us a exact symplectic structure on T^*M. We call it the canonical symplectic structure. Sometimes we would omit the subscript M if no confusion.

3. Let $(X_i, \omega_i), i = 1, 2$ be two symplectic manifolds. One can check by definition, $\omega_1 \times \omega_2 = p_1^*\omega_1 + p_2^*\omega_2$ is a symplectic form on $X_1 \times X_2$. In particular, for a symplectic manifold (X, ω), $X \times \overline{X}$ is also a symplectic manifold.

Because of non-degeneracy of ω, there is a linear isomorphism:

$$\mathcal{X}(X) \xrightarrow{\cong} \Omega^1(X), \qquad V \mapsto \iota_V \omega.$$

Then we can use it to define

Definition 2.4 (Vector fields)**.** For a vector field $V \in \mathcal{X}(X)$.

1. We say V is symplectic if $\iota_V\omega$ is closed.

2. We say V is Hamiltonian if $\iota_V\omega$ is exact. In this case, we say $V = V_H$ if $\iota_{V_H} = -dH$, where $H \in C^\infty(X)$.

For a vector field V, by Cartan formula $\mathcal{L}_V = d\iota_V + \iota_V d$, it is a symplectic vector field if and only if $\mathcal{L}_V\omega = 0$. Moreover, if ϕ_s is the flow of V, we have $\phi_s^*\omega = \omega$. So we define

Definition 2.5 (Morphisms)**.** Let $(X, \omega), (X', \omega')$ be two symplectic manifolds, $\varphi : X \to X'$ be a diffeomorphism.

1. We say φ is a symplectomorphism, if $\varphi^*\omega' = \omega$. The group $\mathrm{Symp}(X, \omega) = \{\varphi \in \mathrm{Diff}(X) : \varphi^*\omega = \omega\}$ is the symplectomorphism group of (X, ω).

2. For an isotopy of symplectomorphism $\varphi_s : X \to X, \varphi_0 = \mathrm{id}_X$, it admits a generator V_s, i.e. there is $\frac{d\varphi_s}{ds} = V_s \circ \varphi_s$. Then V_s is a smooth family of symplectic vector field. If moreover there is $V_s = V_{H_s}$, where $H \in C^\infty(X \times \mathbb{R}), H_s = H(-, s)$. We say φ_s is a Hamiltonian isotopy.

3. If there is a Hamiltonian isotopy φ_s such that $\varphi = \varphi_1$. We say φ a Hamiltonian morphism. One can show $\mathrm{Ham}(X, \omega) = \{\varphi \in \mathrm{Diff}(X) : \varphi \text{ is a Hamiltonian morphism}\}$ is a subgroup of $\mathrm{Symp}(X, \omega)$, we say it the Hamiltonian group.

Remark 2.6. 1. $\mathrm{Symp}(X, \omega)$, and $\mathrm{Ham}(X, \omega)$ both admit a compact support version, denote them by $\mathrm{Symp}^c(X, \omega)$, and $\mathrm{Ham}^c(X, \omega)$.

2. If $H^1_{dR}(X) = 0$, then isotopies of symplectomorphism are Hamiltonian isotopies.

Example 2.7. 1. Consider $\mathbb{R}^{2n}$ with the linear symplectic structure. Then linear symplectomorphism $\mathrm{Sp}(2n, \mathbb{R}) = \{\varphi \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2n}) : \varphi^*\omega = \omega\}$ are tautologically symplectomorphism. Moreover, they are Hamiltonian morphisms.

2. Let M be a smooth manifold, $X = T^*M$ with the canonical symplectic structure. Let $\psi \in \mathrm{Diff}(M)$, consider

$$\Psi : T^*M \to T^*M, (x, p) \mapsto (\psi(x), (df_x^*)^{-1}(p)).$$

Then it is a symplectomorphism. In fact, one can check $\Psi^*\alpha_M = \alpha_M$.

Finally, we need to discuss some submanifolds of symplectic manifolds.

Definition 2.8. Let (X, ω) be a symplectic manifold, $i : S \to X$ be a immersion (embedding). i^*TX is a symplectic vector bundle over S, TS is a sub-bundle of it. There is a symplectic orthogonal complement $TS^{\perp} = \{V \in i^*TX : \omega(V, W) = 0, \forall W \in TS\}$.

1. We say i is a symplectic immersion (embedding), if $TS \subset i^*TX$ is a sub-symplectic vector bundle.

2. We say i is an isotropic immersion (embedding), if $TS \subset TS^{\perp}$, i.e. $i^*\omega = 0$.

3. We say i is a coisotropic immersion (embedding), if $TS^{\perp} \subset TS$.

4. We say i is a Lagrangian immersion (embedding), if $TS^{\perp} \subset TS$.

i is Lagrangian $\iff$ i both isotropic and coisotropic $\iff$ $i^*\omega = 0$, and $2\dim_{\mathbb{R}} S = \dim_{\mathbb{R}} X$.

Example 2.9. 1. Consider $\mathbb{R}^{2n}$ with the linear symplectic structure. $W \subset \mathbb{R}^{2n}$ is a linear subspace. In this case, $TW = W \times W^*$, $TW^{\perp} = W \times W^{\perp}$, so

$$W \text{ is a symplectic embedding if and only if } W \text{ is a symplectic subspace}$$
$$W \text{ is an isotropic embedding if and only if } W \text{ is an isotropic subspace}$$
$$W \text{ is a coisotropic embedding if and only if } W \text{ is a coisotropic subspace}$$
$$W \text{ is a Lagrangian embedding if and only if } W \text{ is a Lagrangian subspace}$$

2. Let M be a smooth manifold, $X = T^*M$ with the canonical symplectic structure.

i) Take a 1-form $\gamma : M \to T^*M$. Then its graph is a Lagrangian submanifold.

ii) If $S \subset M$ is a submanifold, then its conormal bundle $T^*_S M$ is a Lagrangian submanifold.

3. Let (X, ω) be a symplectic manifold, $\varphi : X \to X$ be a diffeomorphism. Consider the graph $\mathrm{Graph}(\varphi) = \{(x, \varphi(x)) : x \in X\}$. Then

$$(-\omega) \times \omega \mid_{\mathrm{Graph}(\varphi)} = \varphi^*\omega - \omega.$$

So

$$\varphi \text{ is a symplectomorphism} \iff \mathrm{Graph}(\varphi) \text{ is a Lagrangian submanifold of } \overline{X} \times X.$$

2.2 Tamarkin's cone trick In the rest of the paper, we will only consider the cotangent bundle T^*M with the canonical symplectic structure. In this case, there is a $\mathbb{R}_+$-action given by $\lambda.(x, p) = (x, \lambda p)$, $\lambda \in \mathbb{R}_+$. We say a subset of cotangent bundle is conic if it is invariant under the $\mathbb{R}_+$-action. We will see later that microsupport of sheaves are conic. However, if we want to deal with non-conic symplectic problems, we need some method to translate them into the conic world.

Here, we will present a method proposed by Tamarkin [Tam13], which we call the cone trick. In fact it is a easy but effective trick. We only need to add a variable $t \in \mathbb{R}$, and denote $(t, \tau) \in T^*\mathbb{R}$. Consider

$$\begin{aligned}
&\dot{T}^*(M \times \mathbb{R}_t) \coloneqq T^*(M \times \mathbb{R}_t) \setminus 0_{M\times\mathbb{R}_t},\\
&T^*_{\tau>0}(M \times \mathbb{R}_t) \coloneqq \{(x, t, p, \tau) \in T^*(M \times \mathbb{R}_t) : \tau > 0\}.
\end{aligned}$$

They are open submanifolds of $T^*(M\times\mathbb{R}_t)$, so they are exact symplectic manifolds. Define the cone map

$$\rho : T^*_{\tau>0}(M \times \mathbb{R}_t) \to T^*M, \quad (x, t, p, \tau) \mapsto (x, p/\tau).$$

One can see, ρ is the symplectic reduction of $T^*_{\tau>0}(M \times \mathbb{R}_t)$ by the coisotropic submanifold $\{\tau = 1\}$. In particular, for $A \subset T^*_{\tau>0}(M\times\mathbb{R}_t)$, we call $\mathrm{Red}(A) \coloneqq \rho(A)$. Moreover, let's introduce the cone associated to $C \subset T^*M$, which is a conic closed subset $\widehat{C} \coloneqq \overline{\rho^{-1}(C)} \subset T^*(M \times \mathbb{R}_t)$.

ρ is very helpful when dealing with symplectic geometry by sheaf method. Here, we present one application about Hamiltonian isotopies.

Proposition 2.10. *[GKS12, Proposition A.6]Let $\phi : T^*M\times I \to T^*M$ be a compact support Hamiltonian isotopy, whose Hamiltonian function is $H \in C^\infty(T^*M \times I)$.*

There is a $\mathbb{R}_+$-equivariant Hamiltonian isotopy $\widehat{\phi} : \dot{T}^*(M \times \mathbb{R}_t) \times I \to \dot{T}^*(M \times \mathbb{R}_t)$ such that

a) $\widehat{H} = \tau H(\rho(-), -)$ is a Hamiltonian function of $\widehat{\phi}$.

b) The diagram is commutative

$$\begin{array}{ccc}
T^*_{\tau>0}(M\times\mathbb{R}_t)\times I & \xrightarrow{\widehat{\phi}} & T^*_{\tau>0}(M\times\mathbb{R}_t)\\
\downarrow{\scriptstyle \rho\times\mathrm{id}_I} & & \downarrow{\scriptstyle \rho}\\
T^*M\times I & \xrightarrow{\phi} & T^*M.
\end{array}$$

c) Moreover, we can take

$$\begin{aligned}
&\widehat{\phi}(x, t, p, \tau, s) = (\tau.\phi(x, p/\tau, s), t + u(x, p/\tau, s), \tau), && \tau \neq 0,\\
&\widehat{\phi}(x, t, p, 0, s) = (x, p, t + v(s), 0), && \tau = 0,
\end{aligned}$$

where $u \in C^\infty(T^*M \times I), v \in C^\infty(I)$.

We call this $\widehat{\phi}$ *or* $\widehat{\phi}_s$ *the conification of* ϕ.

Proof. In fact, it is easy to see, if $\widehat{\phi}$ is given by formula in c), then $\widehat{\phi}$ is $\mathbb{R}_+$-equivariant and the diagram in b) is commutative. Then we only need to check there is a smooth function u such that $\widehat{\phi}$ is a Hamiltonian isotopy with Hamiltonian function $\widehat{H}$.

Let $p : T^*M \times \dot{T}^*\mathbb{R}_t \to \dot{T}^*\mathbb{R}_t, (x, p, t, \tau) \mapsto (t, \tau)$. Consider

$$\psi = \rho \times p : T^*M \times \dot{T}^*\mathbb{R}_t \to T^*M \times \dot{T}^*\mathbb{R}_t, (x, t, p, \tau) \mapsto (x, p/\tau, t, \tau).$$

It is a diffeomorphism. In particular, its tangent map induces an isomorphism between tangent spaces at $q = (x, p, t, \tau) \in T^*M \times \dot{T}^*\mathbb{R}_t$:

$$d\psi_q : T_q(T^*M \times \dot{T}^*\mathbb{R}_t) \xrightarrow{\cong} T_{(x,p/\tau)}(T^*M) \oplus T_{(t,\tau)}(\dot{T}^*\mathbb{R}_t).$$

Let $V_{\widehat{H}}$ be the Hamiltonian vector field given by $\iota_{V_{\widehat{H}}}\omega_{M\times\mathbb{R}} = -d\widehat{H}$. One can decompose $V_{\widehat{H}}$ by above isomorphism, say

$$V_{\widehat{H}} = V_M + V_{\mathbb{R}}.$$

Recall, one can rewrite the exact symplectic structure of $T^*M \times \dot{T}^*\mathbb{R}_t$ into the following form:

$$\begin{aligned} \alpha_{M\times\mathbb{R}} &= \tau\rho^*\alpha_M + p^*\alpha_{\mathbb{R}}, \\ \omega_{M\times\mathbb{R}} &= \tau\rho^*\omega_M + p^*\omega_{\mathbb{R}} + d\tau \wedge \rho^*\alpha_M. \end{aligned}$$

Then

$$\begin{aligned} \iota_{V_{\widehat{H}}}\omega_{M\times\mathbb{R}} =& \tau\iota_{V_M}\rho^*\omega_M + \iota_{V_M}p^*\omega_{\mathbb{R}} + \iota_{V_M}(d\tau \wedge \rho^*\alpha_M) \\ &+ \tau\iota_{V_{\mathbb{R}}}\rho^*\omega_M + \iota_{V_{\mathbb{R}}}p^*\omega_{\mathbb{R}} + \iota_{V_{\mathbb{R}}}(d\tau \wedge \rho^*\alpha_M) \\ =& \left(\tau\iota_{V_M}\rho^*\omega_M + (\iota_{V_{\mathbb{R}}}d\tau)\rho^*\alpha_M\right) + \left(\iota_{V_{\mathbb{R}}}p^*\omega_{\mathbb{R}} - (\iota_{V_M}\rho^*\alpha_M)d\tau\right). \end{aligned}$$

On the other hand, $\widehat{H} = \tau H(\rho(-), -)$, so

$$d\widehat{H}_s = \tau\rho^*dH_s + \rho^*H_s d\tau.$$

However, $\iota_{V_{\widehat{H}}}\omega_{M\times\mathbb{R}} = -d\widehat{H}$ by definition. Comparing them we have

$$-\rho^* dH_s = \iota_{V_M}\rho^*\omega_M + \tau^{-1}(\iota_{V_\mathbb{R}} d\tau)\rho^*\alpha_M,$$
$$-\rho^* H_s d\tau = \iota_{V_\mathbb{R}} p^*\omega_\mathbb{R} - (\iota_{V_M}\rho^*\alpha_M)d\tau.$$

Notice the second equality does not contain dt term, so $V_\mathbb{R} = f\frac{\partial}{\partial t}$, where f is a smooth function. Consequently, $\iota_{V_\mathbb{R}} d\tau = 0$. Therefore, the first equality tells us $V_M = V_{H_s}$. And $f = \rho^*(H_s - \alpha_M(V_{H_s}))$ by the second equality. Then

$$V_{\widehat{H}_s} = V_{H_s} + \rho^*(H_s - \alpha_M(V_{H_s}))\frac{\partial}{\partial t}.$$

In particular, if we take

$$u(x,p,s) = S_H(x,p,s) = \int_0^s (H_\sigma - \alpha_M(V_{H_\sigma}))(\phi_\sigma(x,p))d\sigma,$$

One can check $V_{\widehat{H}_s}$ is the generator of the isotopy $\widehat{\phi}_s$.

It means $\widehat{\phi}_s$ is a Hamiltonian isotopy with Hamiltonian function $\widehat{H}$.

So far, we only define $\widehat{\phi}$ when $\tau \neq 0$. If $\tau = 0$, noticed ϕ is compact support, then H_s and $H_s - \alpha_M(V_{H_s})$ are constants outside of a compact set. Let $v(s) \coloneqq H_s(x) - \alpha_M(V_{H_s})(x)$, x is outside of a compact set. So we could take this v and use formula in c) to extent $\widehat{\phi}$, which is still a Hamiltonian isotopy. □

We have known in the example 2.9-3, that, a diffeomorphism of a symplectic manifold X is a symplectomorphism if and only if its graph is a Lagrangian submanifold of $\overline{X} \times X$. In the spirit of example 2.9-3, we associate $\widehat{\phi}_s$ with a conic Lagrangian submanifold of $T^*_{\tau>0}(M \times \mathbb{R}_t \times I)$, which we call a extended graph of $\widehat{\phi}_s$, say

$$\Lambda_{\widehat{\phi}} \coloneqq \left\{(\widehat{\phi}_s(x,t,p,\tau), (x,t,-p,-\tau), (s, -\widehat{H_s}\circ\widehat{\phi}_s(x,t,p,\tau))) : (x,t,p,\tau)) \in T^*_{\tau>0}(M\times\mathbb{R}_t), s \in I\right\}.$$

One can show it is indeed a conic Lagrangian submanifold [GKS12, Lemma A.1]. Moreover,

$$\pi : T^*_{\tau>0}(M \times \mathbb{R}_t \times I) \to T^*_{\tau>0}(M\times\mathbb{R}_t) \times I,$$

restrict to an injection on $\Lambda_{\widehat{\phi}}$, whose image is the graph of $\widehat{\phi}$. This is why we call it the extended graph.

3 Microlocal Sheaf Theory

We only consider sheaves of k-vector space over M, where k is a field, M is a smooth manifold. In general, we do not make assumption on $\mathrm{char}(k)$, until the end of §4 . Also, the whole theory could be built if k is a commutative ring of finite global dimension [KS90]. Recently, X. Jin and D. Treumann have developed the microlocal sheaf theory into sphere spectrum coefficient, see [JT17, Jin19].

In this section, all complexes of sheaves are locally bounded, i.e. they are bounded on any relatively compact open sets.

Microsupport is well defined on these complexes. 6-operations are almost well posed. In fact, the only subtlety is categories of locally bounded complexes of sheaves are not close under f_*. But in this paper, we only essentially use f_* in cases satisfying $f_* = f_!$, hence we will not deal with the trouble carefully.

All derived categories are full subcategories of unbounded derived categories consisting of locally bounded complexes of sheaves. By abuse of notion, we call the full subcategories $D(M)$.

3.1 Microsupport and functorial estimates [KS82,KS85,KS90] For locally closed subset $j : Z \subset M$, and $F \in D(M)$, we define

$$F_Z := j_! j^{-1} F, \quad R_Z F := Rj_* j^! F.$$

Definition 3.1. [KS90, Definition 5.1.2]Let $F \in D(M)$, $(x, p) \in T^*M$.

We say F does not propagate at (x, p) if $\exists \varphi \in C^1(M), \varphi(x) = 0, d\varphi(x) = p$, such that $R\Gamma_{\{\varphi \geq 0\}}(F)_x \neq 0$.
We call

$$SS(F) := \overline{\{(x, p) \in T^*M : F \text{ does not propagate at } (x, p)\}}$$

the microsupport (or singular support) of F.

There are some basic properties of microsupport:

- Microsupport is a conic closed subset of T^*M.
- $SS(F) \cap 0_M = \pi_M(SS(F) = \mathrm{supp}(F)$ if we identify M with zero section 0_M, where π_M is the cotangent projection.
- The microsupport satisfies the triangular inequality: If there is a distinguished

triangle: $F_1 \to F_2 \to F_3 \xrightarrow{+1}$. Then for $a, b, c \in \{1, 2, 3\}$, we have

$$SS(F_a) \subset SS(F_b) \cup SS(F_c), b \neq c$$
$$SS(F_a) \Delta SS(F_b) \subset SS(F_c), c \neq a, b.$$

Example 3.2. 1. If $\mathcal{L}$ is a non-zero locally constant sheaf on M, then $SS(\mathcal{L}) = 0_M$. In fact one can show the converse is true in the sense [KS90, Theorem 5.4.5(ii)(c)] if $SS(F) = 0_M$, then every cohomology sheaves $\mathcal{H}^i(F)$ are locally constant.

2. If $i : Z \subset M$ is a closed inclusion with smooth boundary. Let $N_i^* = \{(x, -sv(x)) : x \in \partial D, s \geq 0\}$ be the interior conormal bundle of ∂Z, where v is the exterior normal vector field on ∂Z. Then $SS(k_Z) = N_i^*$.

3. If $j : U \subset M$ is an open inclusion with smooth boundary ∂U. Let $N_e^* = \{(x, sv(x)) : x \in \partial D, s \geq 0\}$ be the exterior conormal bundle of ∂U. Then $SS(k_U) = N_e^*$.

4. If $i : S \subset M$ is a closed submanifold. $k_S = i_* i^{-1} k_M \in D(M)$. Then $SS(k_U) = T_S^* M$ is the conormal bundle of S.

5. Let $(X, \mathcal{O}_X)$ be a complex manifold, $\mathcal{M}$ be a coherent D-module, i.e. a module over the ring sheaf $\mathcal{D}_X = \mathrm{Der}(\mathcal{O}_X)$ of holomorphic differential operators. Let $F = R\mathcal{H}om(\mathcal{M}, \mathcal{O}_X)$ be the solution complex, then Kashiwara and Schapira show $SS(F) = char(\mathcal{M})$ [KS90, Theorem 11.3.3], the characteristic of $\mathcal{M}$.

These examples illustrate that the microsupport inherits some geometric information from the symplectic structure of cotangent bundle. In fact, Kashiwara-Schapira prove the following:

Theorem 3.3. *[KS90, Theorem 6.5.4] Let $F \in D(M)$, Then $SS(F) \subset T^*M$ is coisotropic.*

In fact, they show, if X is real analytic, $SS(F)$ being Lagrangian is equivalent to F is constructible in some sense. Here, we need to notice that $SS(F)$ is not smooth everywhere. On smooth points, geometric notions like coisotropic and Lagrangian are well-defined, while on non-smooth points, they could be defined using Whitney's normal cone. Here we will not discuss these facts too much, you can ref [KS90, §6, §8].

Next, let us introduce estimate of microsupport under 6-operations.

Let $f : M \to N$ be a C^∞ map of manifolds. Then there is a diagram of cotangent map:

$$
\begin{array}{ccccc}
T^*M & \xleftarrow{df^*} & M \times_N T^*N & \xrightarrow{f_\pi} & T^*N \\
& \searrow^{\pi} & \downarrow \pi & & \downarrow \pi \\
& & M & \xrightarrow{f} & N
\end{array}
$$

Definition 3.4. Let $f : M \to N$ be a C^∞ map of manifolds, and $\Lambda \subset T^*N$ be a conic subset. One say f is non-characteristic for Λ if

$$(y, p) \in \Lambda, \text{ and } df_y^*(p) = 0 \Rightarrow p = 0.$$

In fact, f is non-characteristic for Λ if and only if $df^* : M \times_N T^*N \to T^*M$ is proper on $f_\pi^{-1}(\Lambda)$. In this case, $df^* f_\pi^{-1}(\Lambda)$ is closed and conic in T^*M.

Theorem 3.5. *[KS90, Theorem 5.4]Let $f : M \to N$ be a C^∞ map of manifolds, $F \in D(N), G \in D(N)$. Let $\omega_{M/N} = f^! k_N$ be the dualizing complex.*

1. *One has*

$$SS(F \overset{L}{\boxtimes} G) \subset SS(F) \times SS(G),$$
$$SS(R\mathcal{H}om(p_M^{-1}F, p_N^{-1}G) \subset SS(F)^a \times SS(G),$$

   where $M \overset{p_M}{\longleftarrow} M \times N \overset{p_N}{\longrightarrow}$.

2. *Assume f is proper on* $\operatorname{supp}(F)$*, then $SS(Rf_!F) \subset f_\pi (df^*)^{-1} (SS(F))$.*

3. *Assume f is non-characteristic for $SS(G)$. Then the natural morphism $f^{-1}G \otimes \omega_{M/N} \to f^!G$ is an isomorphism, and $SS(f^{-1}G) \cup SS(f^!G) \subset df^* f^{-1} (SS(G))$.*

4. *Assume f is submersion. Then $SS(F) \subset M \times_N T^*N$ if and only if $\forall j \in \mathbb{Z}$, the sheaves $\mathcal{H}^j(F)$ are locally constant on the fibres of f.*

Corollary 3.6. *Let $F_1, F_2 \in D(M)$.*

1. *Assume $SS(F_1) \cap SS(F_2)^a \subset 0_M$, then $SS(F_1 \overset{L}{\otimes} F_2) \subset SS(F_1) + SS(F_2)$.*

2. *Assume $SS(F_1) \cap SS(F_2) \subset 0_M$, then $SS(R\mathcal{H}om(F_2, F_1)) \subset SS(F_2)^a + SS(F_2)$.*

In particular, one can deduce the following Microlocal morse lemma by theorem 3.5,

Corollary 3.7. *Let $F \in D^b(M)$, $h \in C^1(M)$, and h is proper on* $\operatorname{supp}(F)$*. Denote $M_t = \{x \in M : h(x) < t\}$.*

For $a, b \in \mathbb{R}$, $a < b$, if $dh(x) \notin SS(F)$ for $a \le h(x) < b$, the restriction morphism $R\Gamma(M_a, F) \to R\Gamma(M_b, F)$ is an isomorphism.

3.2 Compositions and Convolutions [**KS90, §3.6**] [**Tam13, §3**] The composition and convolution are most important sheaf operation in the following geometric application. Let $M_i, i = 1,2,3,4$ be smooth manifolds, V be a $\mathbb{R}$-vector space of fixed dimension $d \in \mathbb{Z}_{>0}$. Throughout this section, we will use notation $M_{1234} = M_1 \times M_2 \times M_3 \times M_4$, $M_{123} = M_1 \times M_2 \times M_3$, $M_{st} = M_s \times M_t$, $s,t = 1,2,3$.

Let's introduce composition first. Consider the diagram:

$$\begin{array}{ccccc} & & M_{123} & & \\ & \swarrow^{p_{12}} & \downarrow^{p_{13}} & \searrow^{p_{23}} & \\ M_{12} & & M_{13} & & M_{23} \end{array}$$

All arrows in the diagram are natural projections.

Composition is a bifunctor:

$$\begin{aligned} \underset{M_2}{\circ} :& D(M_{12} \times V) \times D(M_{23} \times V) \to D(M_{13} \times V), \\ & (F_1, F_2) \mapsto Rp_{13!}(p_{12}^{-1} F_1 \otimes p_{23}^{-1} F_2). \end{aligned}$$

Remark 3.8. Sometimes, we also use subscript $\underset{x_2}{\circ}$, $x_2 \in M_2$. If there is no confusion, we could omit the subscript.

Example 3.9. As corollaries of the proper base change formula and the projection formula. One can shows:

1. For four manifolds M_i, $i = 1,2,3,4$, $F_j \in D(M_{j,j+1})$, $j = 1,2,3$. Then

$$\left(F_1 \underset{M_2}{\circ} F_2\right) \underset{M_3}{\circ} F_3 \cong F_1 \underset{M_2}{\circ} \left(F_2 \underset{M_3}{\circ} F_3\right) \cong Rp_{14!}(p_{12}^{-1} F_1 \otimes p_{23}^{-1} F_2 \otimes p_{34}^{-1} F_3),$$

where

$$p_{st} : M_{1234} \to M_{st}, \quad s,t = 1,2,3,4 \text{are natural projections.}$$

2. Let $M_1 = M_2 = M$, $M_3 = N$. $\Delta \subset M \times M$ be the diagonal. Then

$$k_\Delta \underset{M}{\circ} F \cong F,\ F \in D(M \times N).$$

3. If we identify $D(M \times N)$ and $D(N \times M)$ by $c : M \times N \to N \times M, (m,n) \mapsto (n,m)$. Then for $(F_1, F_2) \in D(M_{12}) \times D(M_{23})$, there is

$$F_1 \underset{M_2}{\circ} F_2 \cong F_2 \underset{M_2}{\circ} F_1.$$

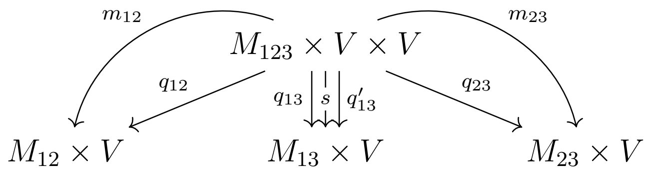
4. Let $M_1 = M$, $M_2 = N$, $M_3 = \{\text{pt}\}$, $n \in N$, and $k_{\{n\}} \in D(N)$ be the skyscraper sheaf. Then

$$F \underset{N}{\circ} k_{\{n\}} \cong F|_{M\times\{n\}}.$$

Remark 3.10. All isomorphisms above are natural. Moreover, we can verify some compatible conditions, which makes $(D(M \times M), \circ, k_\Delta)$ to be a symmetric monoidal category.

Next, let introduce convolution.

Consider the diagram:



Where maps are given as follows:

$$\begin{cases} q_{12}(x_1,x_2,x_3,v,w) &= (x_1,x_2,v),\\ s(x_1,x_2,x_3,v,w) &= (x_1,x_3,v+w),\\ q_{23}(x_1,x_2,x_3,v,w) &= (x_2,x_3,w).\end{cases}$$

$$\begin{cases} m_{12}(x_1,x_2,x_3,v,w) &= (x_1,x_2,v-w),\\ q_{13}(x_1,x_2,x_3,v,w) &= (x_1,x_3,v),\\ q_{23}(x_1,x_2,x_3,v,w) &= (x_2,x_3,w).\end{cases}$$

$$\begin{cases} q_{12}(x_1,x_2,x_3,v,w) &= (x_1,x_2,v),\\ q'_{13}(x_1,x_2,x_3,v,w) &= (x_1,x_3,w),\\ m_{23}(x_1,x_2,x_3,v,w) &= (x_2,x_3,w-v).\end{cases}$$

Convolution over V is a bifunctor:

$$\begin{aligned} \overset{V}{\underset{M_2}{\star}} : D(M_{12}) \times D(M_{23}) &\to D(M_{13}),\\ (F_1,F_2) &\mapsto Rs_!(q_{12}^{-1}F_1 \otimes q_{23}^{-1}F_2),\\ &\cong Rq_{13!}(m_{12}^{-1}F_1 \otimes q_{23}^{-1}F_2),\\ &\cong Rq'_{13!}(q_{12}^{-1}F_1 \otimes m_{23}^{-1}F_2).\end{aligned}$$

Remark 3.11. 1. Sometimes, we also use subscript $\overset{v}{\underset{x_2}{\star}}$, $v \in V, x_2 \in M_2$. For simplify, we could omit the superscript and subscript if there is no confusion.

2. In some cases, convolution could be presented by composition. But, sometimes, convolution is working on spaces of lower dimension. Hence, we prefer to use convolution in this paper.

3. There is a non-proper convolution, denote by $\star_{np}$, which is defined by just replace proper direct image by direct image in the definition. We only use it in the microlocal cut-off lemma introduced soon.

The microlocal cut-off lemma provides us a functorial way to cut-off the microsupport.

Theorem 3.12. *[KS90, Proposition 3.5.4, 5.2.3][Microlocal cut-off lemma]Let M be a smooth manifold. $\gamma \subset V = \mathbb{R}^n$ be a close convex cone with vertex at 0. Then, for $F \in D(M \times V)$,*

$$SS(F) \subset T^*M \times \mathbb{R}^d \times \gamma^\circ \iff k_\gamma \overset{V}{\star}_{np} F \cong F.$$

Here the convolution is taking over $M_1 = M_2 = \{pt\}$, $M_3 = M$, $V = \mathbb{R}^d$, and $\gamma^\circ := \{p \in V^* : \langle x, p\rangle \geq 0, \forall x \in \gamma\}$.

Example 3.13. Similar to the case of composition, the proper base change formula and the projection formula show

1. For four manifolds M_i, $i = 1,2,3,4$, $F_j \in D(M_{j,j+1} \times V)$, $j = 1,2,3$, $F_4 \in D(M_{34})$. Then in $D(M_{14} \times V)$,

$$\left(F_1 \underset{M_2}{\star} F_2\right) \underset{M_3}{\star} F_3 \cong F_1 \underset{M_2}{\star} \left(F_2 \underset{M_3}{\star} F_3\right) \cong Rs_!(q_{12}^{-1}F_1 \otimes q_{23}^{-1}F_2 \otimes q_{34}^{-1}F_3),$$
$$\left(F_1 \underset{M_2}{\star} F_2\right) \underset{M_3}{\circ} F_4 \cong F_1 \underset{M_2}{\star} \left(F_2 \underset{M_3}{\circ} F_4\right) \cong R\sigma_!(q_{12}^{-1}F_1 \otimes q_{23}^{-1}F_2 \otimes p_{34}^{-1}F_4),$$

where

$$\begin{aligned}
s :& M_{1234} \times V_1 \times V_2 \times V_3 \to M_{14} \times V,\\
& (x_1, x_2, x_3, x_4, v_1, v_2, v_3) \mapsto (x_1, x_4, v_1 + v_2 + v_3),\\
\sigma :& M_{1234} \times V_1 \times V_2 \to M_{14} \times V,\\
& (x_1, x_2, x_3, x_4, v_1, v_2) \mapsto (x_1, x_4, v_1 + v_2),\\
q_{j,j+1} :& M_{1234} \times V_1 \times V_2 \times V_3 \to M_{j,j+1} \times V_j,\ j = 1,2,3,\\
p_{34} :& M_{1234} \times V_1 \times V_2 \to M_{34}.
\end{aligned}$$

2. Let $M_1 = M_2 = M$, $M_3 = N$. $\Delta \subset M \times M$ be the diagonal. Then for any $F \in D(M \times N \times V)$

$$k_{\Delta\times\{0\}} \underset{M}{\star} F \cong F.$$

3. If we identify $D(M \times N \times V) \cong D(N \times M \times V)$ by $c : M \times N \to N \times M, (m, n) \mapsto (n, m)$. Then for any $(F_1, F_2) \in D(M_{12} \times V) \times D(M_{23} \times V)$, there is

$$F_1 \underset{M_2}{\star} F_2 \cong F_2 \underset{M_2}{\star} F_1.$$

4. Let $M_1 = M$, $M_2 = N$, $M_3 = \{\mathrm{pt}\}$, $n \in N$, and $k_{\{n\}} \in D(N)$ be the skyscraper sheaf. Then for any $F \in D(M \times N \times V)$

$$F \underset{N}{\circ} k_{\{(n,0)\}} \cong F|_{M\times\{n\}\times V}.$$

5. Let $M_1 = M$, $M_2 = M_3 = \{\mathrm{pt}\}$, $c \in V$,

$T_c : M \times V \to M \times V, (m, v) \mapsto (m, v + c)$. Then for any $F \in D(M \times V)$

$$F \star k_{\{c\}} \cong T_{c*}F.$$

Remark 3.14. We also remark, all isomorphisms here are natural. Moreover, we can verify some compatible conditions, which makes $\left(D(M \times M \times V), \overset{V}{\star}, k_{\Delta\times\{0\}}\right)$ to be a symmetric monoidal category.

3.3 Tamarkin Category [**Tam13, GS14**] Before introducing Tamarkin Category, let us review notions of projector and semi-orthogonal decomposition of triangulated categories. One can see [KS06, §4.1, Exercises 10.15] for more details.

Definition 3.15. Let $\mathcal{C}$ be a category. A projector (P, ε) is a pair, where $P : \mathcal{C} \to \mathcal{C}$ is a functor, and $\varepsilon : \mathrm{id}_{\mathcal{C}} \Rightarrow P$ is a morphism of functors such that $\varepsilon \circ P$, $P \circ \varepsilon : P \Rightarrow$ are isomorphisms.

Proposition 3.16. *Let (P, ε) be a projector on $\mathcal{C}$.*

1. *For any $X, Y \in \mathcal{C}$, the morphism*

$$\mathrm{Hom}_{\mathcal{C}}(P(X), P(Y)) \xrightarrow{\circ\varepsilon_X} \mathrm{Hom}_{\mathcal{C}}(X, P(Y)),$$

is bijective.

2. *Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ consisting objects satisfying $\varepsilon_X : X \to P(X)$*

is an isomorphism. Then $P(X) \in \mathcal{C}_0$ and P induces a functor $\mathcal{C} \to \mathcal{C}_0$ which is left adjoint to the inclusion functor $\iota : \mathcal{C}_0 \to \mathcal{C}$.

Proposition 3.17. *Let $R : \mathcal{C}' \to \mathcal{C}$ be a fully faithful functor and admits a left adjoint $L : \mathcal{C} \to \mathcal{C}'$. Let $\varepsilon :$ are the unit and counit of adjoint the pairs (L, R).*

Let $P = R \circ L$. Then (P, ε) is a projector. Let $\mathcal{C}_0$ be the same full subcategory in proposition 3.16, then $\mathcal{C}' \simeq \mathcal{C}_0$.

These propositions reflect projectors are indeed similar to projectors in linear algebra.

Moreover, in triangulated categories, the notion of projector is related to the notion of semi-orthogonal decomposition.

Proposition 3.18. *Let $\mathcal{T}$ be a triangulated category, $\mathcal{N}$ be a null system, which is a full triangulated subcategory stable under isomorphisms. Consider the full triangulated subcategory, so called left semi-orthogonal complement,*

$$^{\perp}\mathcal{N} := \{X \in \mathcal{T} : \mathrm{Hom}_{\mathcal{T}}(X, Y), \forall Y \in \mathcal{N}\}.$$

If the inclusion $^{\perp}\mathcal{N} \to \mathcal{T}$ admit a left adjoint $L : \mathcal{T} \to^{\perp} \mathcal{N}$, L induces an equivalent $\mathcal{T}/\mathcal{N} \simeq^{\perp} \mathcal{N}$, where $\mathcal{T}/\mathcal{N}$ is the Verdier localization.

Similarly, one can define right semi-orthogonal complement. And correspondence results are still true.

This proposition tells us, in situations of triangulate categories, semi-orthogonal complements could provide us projectors, and conversely, projectors could also provide us semi-orthogonal complements. And they provide some models of Verdier localization.

Example 3.19. Let $\mathcal{A}$ be an Abelson category, $K(\mathcal{A})$ be the homotopy category of (unbounded) complexes in $\mathcal{A}$, $\mathcal{N} \subset K(\mathcal{A})$ be the null system of acyclic complexes. Then $^{\perp}\mathcal{N}$ is the category of hoprojective complexes.

If we assume $\mathcal{A}$ has enough hoprojectives, which means for any complexes X in $\mathcal{A}$, there is a quasi-isomorphism $P \to X$, such that P is a hoprojective complex. Such P is called a hoprojective resolution of X. Then one can shows, taking a hoprojective resolution could build a functor $F : K(\mathcal{A}) \to^{\perp} \mathcal{N}$, which is left adjoint to inclusion functor $\iota :^{\perp} \mathcal{N} \subset K(\mathcal{A})$. And $\iota \circ F : K(\mathcal{A}) \to K(\mathcal{A})$ together with the counit $\mathrm{id} \Rightarrow \iota \circ F$ is a projector. F induces an equivalent $\overline{F} : K(\mathcal{A})/N \to^{\perp} \mathcal{N}$, such that $Q\overline{F} = F$, where $Q : K(\mathcal{A}) \to K(\mathcal{A})/N$ is the localizing functor. In particular,

$K(\mathcal{A})/N = D(\mathcal{A})$ is the unbounded derived category of $\mathcal{A}$, here we present a model of it.

We defined the convolution in the last section. Here, we will see that the convolution provides us some projector and could be used to microlocalization.

Let $\gamma \subset V$, where V is a finite dimensional $\mathbb{R}$ vector space and γ is a closed convex cone. Its polarized cone is $\gamma^\circ := \{p \in V^* : \langle x, p\rangle \geq 0, \forall x \in \gamma\}$. Let

$$L_\gamma = k_\gamma \overset{V}{\star} : D(M \times V) \to D(M \times V),$$

where $M_1 = M_2 = \{\mathrm{pt}\}$, $M_3 = M$. The morphism $k_\gamma \to k_{\{0\}}$ induces a morphism of functors: $\varepsilon : L_\gamma \Rightarrow \mathrm{id}_{D(M\times V)}$. Because $k_\gamma \overset{V}{\star} k_\gamma \cong k_\gamma$ and example 3.13-1, we have (L_γ, ε) is a projector on $D(M \times V)^{op}$.

Set

$$U_\gamma = T^*M \times V \times \mathrm{Int}(\gamma^\circ),$$
$$Z_\gamma = T^*(M \times V) \setminus U_\gamma.$$

Besides, set $D_{Z_\gamma}(M \times V)$ be the full triangulated category of $D(M \times V)$ consist of sheaves satisfying $SS(F) \subset Z_\gamma$, moreover, it is a null system by the triangle inequality of microsupport.

Motivated by work of Tamarkin [Tam13] in the case of $\gamma = [0, \infty) \subset \mathbb{R}$, in [GS14], the authors shows

Theorem 3.20. *[GS14, Proposition 3.19, 3.21]*

1. *Let $F \in D(M \times V)$ then $F \in^\perp D_{Z_\gamma}(M \times V)$ if and only if $L_\gamma F \xrightarrow{\cong} F$.*
2. *Let $G \in D_{Z_\gamma}(M \times V)$, then $L_\gamma G \cong 0$.*

   Consequently, by proposition 3.18,
3. *L_γ functor through localizing functor $Q_\gamma : D(M \times V) \to D(M \times V)/D_{Z_\gamma}(M \times V) =: D(M \times V; U_\gamma)$, and induces a functor $l_\gamma : D(M \times V; U_\gamma) \to D(M \times V)$.*
4. *l_γ is left adjoint to Q_γ, and induces an equivalence $D(M \times V; U_\gamma) \simeq^\perp D_{Z_\gamma}(M \times V)$.*

Consider the functor: $\Psi_\gamma : D(M) \to D(M \times V), F \mapsto F \boxtimes k_\gamma$. One can check that $L_\gamma \circ \Psi_\gamma \cong \Psi_\gamma$, so it induces a functor

$$\overline{\Psi}_\gamma : D(M) \to D(M \times V; U_\gamma).$$

Theorem 3.21. *[GS14, Proposition 3.24]* $\overline{\Psi}$ *is a fully faithful functor.*

The theorem tells us, the category $D(M \times V; U_\gamma) \simeq^\perp D_{Z_\gamma}(M \times V)$ has at least as many objects as $D(M)$.

Now, let us consider a specific case: $V = \mathbb{R}_t$, and $\gamma = [0, \infty)$. If we set $V^* = \mathbb{R}_\tau$, we have $U_\gamma = \{\tau > 0\}$, $Z_\gamma = \{\tau \leq 0\}$.

Definition 3.22. [GS14, §4][Tamarkin Category] $\mathcal{D}(M) :=^\perp \left(D_{\{\tau \leq 0\}}(M \times \mathbb{R}_t)\right) \cong D(M \times \mathbb{R}_t; \{\tau > 0\})$.

theorem 3.20 still holds, in particular, $F \in \mathcal{D}(M) \iff k_{[0,\infty)} \overset{t}{\star} F \cong F$.

Example 3.23. Let $M = \mathbb{R}^d$, $k_Z \in D(M \times \mathbb{R}_t)$, $Z = \{(x_1, \cdots, x_d, t) : |x_\alpha| \leq t, \alpha = 1, \cdots, d\}$ be a close set.

$$k_{[0,\infty)} \overset{t}{\star} k_Z = Rs_!(k_W),$$

where

$$W = \{(x, t_1, t_2) \in M \times \mathbb{R} \times \mathbb{R} : (x, t_1) \in Z, t_2 \geq 0\}$$
$$s : M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R},\ (x, t_1, t_2) \mapsto (x, t_1 + t_2).$$

One can check s restricts to W a proper map with contractible fibres.

Moreover $s(W) = Z$, and $k_{[0,\infty)} \overset{t}{\star} k_Z = Rs_!(k_W) \xrightarrow{\cong} k_Z$. So $k_Z \in \mathcal{D}(M)$.

One can also see that $k_{Z+(x_0,t_0)} \in \mathcal{D}(M)$, $(x_0, t_0) \in M \times \mathbb{R}_t$.

Definition 3.24. The full subcategory $\mathcal{D}^{t+}(M)$ of $\mathcal{D}(M)$ consists of objects which support bounded below by t-direction. i.e.

$$F \in \mathcal{D}^{t+}(M) \iff F \in \mathcal{D}(M) \text{ and } \mathrm{supp}(F) \subset \{t \geq A\}, A > -\infty.$$

Let $t_F := \sup\{A : \mathrm{supp} F \subset \{t \geq A\}\}$.

Noticed, $\overline{\Psi}$ factor through $\mathcal{D}^{t+}(M) \xrightarrow{\iota} \mathcal{D}(M) \simeq D(M \times \mathbb{R}_t; \{\tau > 0\})$.

Another advantage is that the microlocal cut-off lemma theorem 3.12 provides us an easy way to test whether $F \in D(M \times \mathbb{R}_t)$ to be objects of $\mathcal{D}^{t+}(M)$.

Proposition 3.25. *Let* $F \in D(M \times \mathbb{R}_t)$ *, then*

$$F \in \mathcal{D}^{t+}(M) \iff SS(F) \subset \{\tau \geq 0\} \textit{ and } \mathrm{supp}(F) \subset \{t \geq A\}, A > -\infty.$$

Proof. In fact, when $\mathrm{supp}(F) \subset \{t \geq A\}$, one can easily check that

$$k_{[0,\infty)} \overset{t}{\star}_{np} F \cong k_{[0,\infty)} \overset{t}{\star} F$$

The microlocal cut-off lemma claims that

$$k_{[0,\infty)} \overset{t}{\star}_{np} F \iff SS(F) \subset \{\tau \geq 0\}$$

One can conclude by combining these two facts. ☐

3.4 Guillermou-Kashiwara-Schapira Sheaf Quantization [GKS12] In §2.2, we construct a conification $\widehat{\phi}$ of a compact support Hamiltonian isotopy ϕ on cotengent bundle T^*M, whose extent graph $\Lambda_{\widehat{\phi}}$ is a conic Lagrangian submanifold of $T^*_{\tau>0}(M \times \mathbb{R}_t \times I)$. In the spirit of theorem 3.3 and example 2.9-3, one hope to construct a (complex of) sheaf, as a sheaf theoretical replacement of Hamiltonian isotopy. In [GKS12], Guillermou-Kashiwara-Schapira provided us a candidate.

Theorem 3.26. *[GKS12, Theorem 3.7]Let $\varphi : \dot{T}^*N \times I \to \dot{T}^*N$ be a $\mathbb{R}_+$-equivariant Hamiltonian isotopy. There is $\mathcal{K} \in D(N \times N \times I)$, such that*

1) $SS(\mathcal{K}) \subset \Lambda_{\widehat{\phi}} \cup 0_{N\times N\times I}$,

2) $\mathcal{K}_0 = k_{\Delta_N}$.

where $\mathcal{K}_{s_0} = \mathcal{K}|_{\{s=s_0\}}$.

If we set $\mathcal{K}_s^{-1} = v^{-1}R\mathcal{H}om(\mathcal{K}_s, \omega_N \overset{L}{\boxtimes} k_N)$, $v(x,y) = (y,x), x, y \in N$, then

a) $\mathrm{supp}(\mathcal{K}) \rightrightarrows N \times I$ *are both proper,*

b) $\mathcal{K}_s \circ \mathcal{K}_s^{-1} \cong \mathcal{K}_s^{-1} \circ \mathcal{K}_s \cong k_\Delta$,

c) $\mathcal{K}$ *is unique up to unique isomorphism.*

(Here the composition is over $M_1 = M_2 = M_3 = N$).

Because of c), we denote them by $\mathcal{K}(\varphi) := \mathcal{K}$ and $\mathcal{K}(\varphi_s) := \mathcal{K}_s$, $\forall s \in I$, call them the **GKS sheaf quantization** of φ.

Remark 3.27. In fact, in [GKS12], the authors showed that if the parameter interval I if bounded, then $\mathcal{K}(\varphi)$ is a bounded complex. In particular, $\mathcal{K}(\varphi_s)$ are all bounded for all $s \in I$.

It follows $b)$ that

$$\circ\mathcal{K}(\phi_s) : D(M) \to D(M), F \mapsto F \circ \mathcal{K}(\phi_s)$$

induces equivalents of categories. Moreover, functorial estimates theorem 3.5 show

$$\dot{SS}(F \circ \mathcal{K}(\phi_s)) = \varphi_s(\dot{SS}(F)),\ \dot{SS}(F) \coloneqq SS(F) \cap \dot{T}^*M.$$

Now we use $N = M \times \mathbb{R}_t$, and $\varphi = \widehat{\phi}$ for a compact support Hamiltonian isotopy ϕ on T^*M. Because $\circ\mathcal{K}(\widehat{\phi}_s)$ is a composition functor, by example 3.13-1, it induces a functor on Tamarkin categories:

$$\circ\mathcal{K}(\widehat{\phi}_s) : \mathcal{D}(M) \to \mathcal{D}(M).$$

And combining with diagram of proposition 2.10-b), there is

$$\rho\left(\dot{SS}(F \circ \mathcal{K}(\widehat{\phi}_s)\right) = \phi_s\left(\rho(\dot{SS}(F))\right).$$

Moreover, $\circ\mathcal{K}(\widehat{\phi}_s)$ acts on $\mathcal{D}^{t+}(M)$. In fact, proposition 2.10-c) tells us

$$(x', p', t', \tau) = \widehat{\phi}(x, t, p, \tau, s) = (\tau.\phi(x, p/\tau, s), t + S_H(x, p/\tau, s), \tau),$$

where $S_H(x, p, s) = \int_0^s (H_\sigma - \alpha_M(V_{H_\sigma}))(\phi_\sigma(x, p))d\sigma$ is the action function.

So for $F \in \mathcal{D}^{t+}(M)$, $\mathrm{supp}(F) \subset \{t \geq t_F\}$, the formula of $\widehat{\phi}(x, t, p, \tau, s)$ shows

$$\dot{SS}(F \circ \mathcal{K}(\widehat{\phi}_s) \subset \{t' \geq t_F + \min S_H\}.$$

It follows $\pi_M(SS(F)) = \mathrm{supp}(F)$, that $\mathrm{supp}\left(F \circ \mathcal{K}(\widehat{\phi}_s)\right) \subset \{t' \geq t_F + \min S_H\}$, i.e. $F \circ \mathcal{K}(\widehat{\phi}_s) \in \mathcal{D}^{t+}(M)$.

3.5 Sheaf Displacement Energy We have seen in §3.3, objects in Tamarkin category are (complex of) sheaves on $M \times \mathbb{R}_t$. The variable t used for conification is tricky. However, in this section, we will see that, some geometry informations related to symplectic rigidity is inherent in the variable t. We will present a construction relate to it, and then show some simple properties. They will be used to prove some non-squeezing results in §4.

On $M \times \mathbb{R}_t$, there is a diffeomorphism

$$T_c : M \times \mathbb{R}_t \to M \times \mathbb{R}_t, \quad (x,t) \mapsto (x, t+c).$$

Recall, $F \in \mathcal{D}(M) \iff F \star k_{[0,+\infty)} \cong F$. The isomorphism induces morphisms between functors $\tau_c : \mathrm{Id} \Rightarrow T_{c*}$ [Tam13, GS14], for any $c \in \mathbb{R}_{\geq 0}$ as follow:

Closed inclusion $[c, +\infty) \to [0, +\infty)$ induces morphism of sheaves: $i : k_{[0,+\infty)} \to k_{[c,+\infty)}$. Then the convolution, as a functor, induces $F \star k_{[0,+\infty)} \to F \star k_{[c,+\infty)}$. Now τ_c is given by the following diagram, through example 3.13-5,

$$\begin{array}{ccccccc}
k_{[0,+\infty)} \overset{t}{\star} F & & & \xrightarrow{\cong} & & & F \\
\downarrow{\scriptstyle F\star i} & & & & & & \downarrow{\scriptstyle \tau_c(F)} \\
k_{[c,+\infty)} \overset{t}{\star} F & \xrightarrow{\cong} & T_{c*} k_{[0,+\infty)} \overset{t}{\star} F & \xrightarrow{\cong} & T_{c*}(k_{[0,+\infty)} \overset{t}{\star} F) & \xrightarrow{\cong} & T_{c*}F
\end{array}$$

Example 3.28. Let $M = \mathbb{R}^d$, $k_Z \in D(M \times \mathbb{R}_t)$, $Z = \{(x_1, \cdots, x_d, t) : |x_\alpha| \leq t, \alpha = 1, \cdots, d\}$. We computed in example 3.23 that $F = k_{Z+(x_0,t_0)} \in \mathcal{D}(M)$

For $c \geq 0$, $k_{[c,\infty)} \overset{t}{\star} F \cong T_{c*}F = k_{Z+(x_0,t_0+c)}$.

$\tau_c(F) : F \to T_{c*}F$ is exactly given by closed inclusion $Z + (x_0, t_0 + c) \subset Z + (x_0, t_0)$.

Definition 3.29. For $F \in \mathcal{D}(M)$, define the *Sheaf displacement energy* of F:

$$e(F) \coloneqq \inf\{c \geq 0 : \tau_c(F) = 0\}.$$

We say F is a torsion object if $e(F) < +\infty$.

In some situations, we need properness. So we also define the following version of displacement energy.

$$e'(F) \coloneqq \sup\{e(F|_{M\times[a,b)}) : \forall a, b \in \mathbb{R}, a < b\}.$$

Remark 3.30. Asano-Ike [AI17, Definition 4.17], Zhang [Zha18, Definition 4.26] provide some similar constructions.

Proposition 3.31. *For $F \in \mathcal{D}(M)$, $e(F) \geq e'(F)$.*

Proof. First of all, one can prove,

$$F|_{M\times[a,b)} \cong F \circ k_{\Delta_{[a,b)}},$$

where $\Delta_{[a,b)} = \{(x,x) \in \mathbb{R}^2 : a \leq x < b\}$. Then, example 3.13-1 provides us a commutative diagram:

$$\begin{array}{ccccc} T_{c*}(F \circ k_{\Delta_{[a,b)}}) & \xrightarrow{\cong} & k_c \overset{t}{\star} (F \circ k_{\Delta_{[a,b)}}) & \xrightarrow{\cong} & (T_{c*}F) \circ k_{\Delta_{[a,b)}} \\ & {\scriptstyle \tau_c(F \circ k_{\Delta_{[a,b)}})} \searrow & & \swarrow {\scriptstyle \tau_c(F) \circ k_{\Delta_{[a,b)}}} & \\ & & F \circ k_{\Delta_{[a,b)}} & & \end{array}$$

The diagram shows $e(F) \geq e(F|_{M \times [a,b)})$. Consequently, $e(F) \geq e'(F)$. □

Example 3.32. Let $g : [0, \infty) \to [0, \infty)$ be a smooth function. Assume $g'(x) \leq 0$ such that $g(0) > 0$, $g(a) = 0$, and $g'(x) < 0$, $0 < x \leq a$; $g'(x) = 0$ otherwise.

Consider a function $f : \mathbb{R}^d \to [0, \infty)$, $f(x) = g(\|x\|^2)$. Then $f \in C^\infty(\mathbb{R}^d)$.

Now let $Z = \{(x,t) \in \mathbb{R}^d \times \mathbb{R}_t :, -f(x) \leq t < f(x)\}$. Then $k_Z \in \mathcal{D}(\mathbb{R}^d)$.

In fact, let $Z_\pm = \{(x,t) \in \mathbb{R}^d \times \mathbb{R}_t :, t \geq \pm f(x)\}$, then Z fits into the excision distinguished triangle

$$k_Z \to k_{Z_-} \to k_{Z_+} \xrightarrow{+1} .$$

Then the triangular inequality with example 3.2 shows $SS(k_Z) \subset SS(k_{Z_+}) \cup SS(k_{Z_-}) \subset \{\tau \geq 0\}$. Besides, $\operatorname{supp}(k_Z) \subset \{t \geq -f(0)\}$. Then $k_Z \in \mathcal{D}(\mathbb{R}^d)$ by proposition 3.25.

Moreover,

$$\operatorname{Hom}_{\mathcal{D}(\mathbb{R}^d)}(k_Z, T_{c*}k_Z) = \operatorname{Hom}_{D(\mathbb{R}^d \times \mathbb{R}_t)}(k_Z, T_{c*}k_Z) = \begin{cases} k, & 0 \leq c < 2f(0), \\ 0, & c \geq 2f(0). \end{cases}$$

and the induced morphism

$$\operatorname{Hom}_{D(\mathbb{R}^d \times \mathbb{R}_t)}(k_Z, k_Z) \xrightarrow{\circ \tau_c(k_Z)} \operatorname{Hom}_{D(\mathbb{R}^d \times \mathbb{R}_t)}(k_Z, T_{c*}k_Z)$$

is the identity for $0 \leq c < 2f(0)$. Hence, $e(k_Z) = 2f(0) = 2g(0)$.

The following simple proposition is crucial for our application.

Proposition 3.33. *If $K \in \mathcal{D}(M \times M), F \in \mathcal{D}(M)$, and $K \star F \cong F$. Then there is $e(K) \geq e(F)$.*

Proof. We have computed that in example 3.13-5 that there are isomorphisms between functors:

$$T_{c*}(F) \cong F \star k_{\{c\}} \cong k_{\{c\}} \star F, c \in \mathbb{R}$$

Applying these isomorphisms to the identity $K \star F \cong F$ with $c \geq 0$. Combine with example 3.13-1. We obtain a diagram,

$$\begin{array}{ccccccccc}
T_{c*}K \star F & \xrightarrow{\cong} & (k_c \star K) \star F & \xrightarrow{\cong} & k_c \star (K \star F) & \xrightarrow{\cong} & T_{c*}(K \star F) & \xrightarrow{\cong} & T_{c*}F \\
\downarrow{\scriptstyle \tau_c(K) \star F} & & & & & & & & \downarrow{\scriptstyle \tau_c(F)} \\
K \star F & & & & \xrightarrow{\cong} & & & & F
\end{array}$$

The diagram show us, if $\tau_c(K) = 0$, $\tau_c(F) = 0$. It is exactly means that $e(K) \geq e(F)$.

□

If we want to obtain non-squeezing results, we need inputs coming from Hamiltonian isotopies. As we have shown in last section, the best candidate here is the GKS sheaf quantization. Here, let us show sheaf displacement energy is invariant under GKS quantization.

Proposition 3.34. *Let $\phi_s \in Ham_c(T^*M), s \in \mathbb{R}$ be a compact support Hamiltonian isotopy. $K(\widehat{\phi_s}) \in D^{lb}(M \times \mathbb{R}_t \times M \times \mathbb{R}_t)$ is the GKS quantization of the conification of ϕ_s, $s \in \mathbb{R}$.*

Then for all $F \in \mathcal{D}(M)$, $e(F) = e(F \circ K(\widehat{\phi_s}))$, $s \in \mathbb{R}$

Proof. In fact, the GKS quantization functor given by composition

$$K(\widehat{\phi_s}) : \mathcal{D}(M) \to \mathcal{D}(M), F \mapsto F \circ K(\widehat{\phi_s})$$

is a auto-equivalence, whose quasi-inverse is also given by a composition whose kernel is $K(\widehat{\phi_s})^{-1}$.

However, consider the diagram followed example 3.13-1.

$$\begin{array}{ccccccc}
T_{c*}F \circ K(\widehat{\phi_s}) & \xrightarrow{\cong} & (k_c \overset{t}{\star} F) \circ K(\widehat{\phi_s}) & \xrightarrow{\cong} & k_c \overset{t}{\star} (F \circ K(\widehat{\phi_s}) & \xrightarrow{\cong} & T_{c*}(F \circ K(\widehat{\phi_s})) \\
& {\scriptstyle \tau_c(F) \circ K(\widehat{\phi_s})} \searrow & & & & \swarrow {\scriptstyle \tau_c(F \circ K(\widehat{\phi_s}))} & \\
& & F \circ K(\widehat{\phi_s}) & & & &
\end{array}$$

It show us that $e(F) \geq e(F \circ K(\widehat{\phi_s}))$. Applying the argument to $F \circ K(\widehat{\phi_s})$ and $\circ K(\widehat{\phi_s})^{-1}$ shows the inverse inequality. □

4 Square Projector and Non-squeezing

In this section, we shall consider the square projector. It is a projector of Tamarkin category, which project sheaves into sheaves microsupport in the square.

Throughout this section, we set E real vector spaces of dimension d. In §4.1, §4.2, we assume in addition $d = 1$. Also, we set $V = E \times \mathbb{R}_t$.

4.1 Construction of Square Projector Let us consider the full subcategory $\mathcal{D}^{t+}(E)$ of $\mathcal{D}(E)$ defined in definition 3.24 consisting of objects which support bounded below by t-direction.

For square $\square = \{(x,p) \in T^*E : |x| \leq 1, |p| \leq 1\}$. Using the Tamarkin's cone map, there is a cone

$$\widehat{\square} = \overline{\rho^{-1}(\square)} = \{(x,p,t,\tau) \in T^*_{\tau>0}(E \times \mathbb{R}_t) : |x| \leq 1, |p| \leq \tau\}.$$

Motived by the microlocal cut-off lemma theorem 3.12, we define a functor as the building block of our square projector.

Let $\gamma_1 = \{(x,t) \in V : |x| \leq t\}$ be a closed convex cone. Its polarized cone $\gamma_1^\circ = \{(p,\tau) \in V^* : |p| \leq \tau\}$. Under their help, we can see $\widehat{\square} = [-1,1] \times \mathbb{R}_t \times \gamma_1^\circ$.

For $F \in D(E \times \mathbb{R}_t)$, define $G(F) = k_{\gamma_1} \overset{V}{\star} (F_{(-1,1)\times\mathbb{R}_t})$. Besides, there are two morphisms of functors:

1. $F_{(-1,1)\times\mathbb{R}_t} \to F$ induced by open inclusion $(-1,1) \times \mathbb{R}_t \to E \times \mathbb{R}_t$.
2. $k_{\gamma_1} \overset{V}{\star} F \to F$, induced by closed inclusion $\Delta_E \times \{0\} \to \gamma_1$.

Their composition provides us a morphism of functor

$$G \Rightarrow \mathrm{Id},\, G(F) \to F.$$

Proposition 4.1. *$G(F) = K_0 \overset{t}{\star} F$, where*

$$K_0 = k_{W_0}, W_0 = \{(x_2, x_1, t) \in E_2 \times E_1 \times \mathbb{R}_t : |x_2 - x_1| \leq t, |x_1| < 1\}.$$

The morphism of functor $G \Rightarrow Id$ is given by $\phi_{-1} : k_{W_0} \to k_{\Delta_E \times \{0\}}$, which is induced by $W_0 \subset \{(x_2,x_1,t) : |x_2 - x_1| \leq t\} \supset \Delta_E \times \{0\}$.

In particular, G restrict to a endfunctor on $\mathcal{D}^{t+}(E)$, with a morphism of functor $G \Rightarrow Id$.

Proof. G is a composition of two functors, then we could compute their kernels separately, and them compute their convolution. So does these morphism of functors.

To be more clear, let's present maps here in a diagram.

$$
\begin{array}{ccccc}
 & & E_2 \times E_1 \times \mathbb{R}_{t_2} & & \\
 & & \uparrow{\scriptstyle m_{32}^{\mathbb{R}_t}} & & \\
 & & V_2 \times V_1 = E_2 \times E_1 \times \mathbb{R}_{t_2} \times \mathbb{R}_{t_1} & & \\
 & \swarrow{\scriptstyle m_{32}^{V}} & \downarrow{\scriptstyle q_{31}} & \searrow{\scriptstyle q_{21}} & \\
V_2 = E_2 \times \mathbb{R}_{t_2} & & V_2 = E_2 \times \mathbb{R}_{t_2} & & V_1 = E_1 \times \mathbb{R}_{t_1}
\end{array}
$$

1. (Space cut-off)

$$
\begin{aligned}
k_{\Delta_{(-1,1)}\times\{0\}} \overset{t}{\star} F &= Rq_{31!}((m_{12}^{\mathbb{R}_t})^{-1} k_{\Delta_{(-1,1)}\times\{0\}} \otimes q_{32}^{-1} F) \\
&= Rq_{31!}(k_{\{(x_1,x_1,t,t):|x_1|<1\}} \otimes q_{32}^{-1} F) \\
&= Rq_{32!}(k_{\{(x_1,x_1,t,t):|x_1|<1\}} \otimes q_{32}^{-1} F) \\
&= F_{(-1,1)\times\mathbb{R}_t}.
\end{aligned}
$$

The third equality follows that $q_{32} = q_{31}$ on $\{(x_1, x_1, t, t)\}$. Last equality follows directly from the projection formula.

Moreover, the proof reveal that $F_{(-1,1)\times\mathbb{R}_t} \to F$ is given by $k_{\Delta_{(-1,1)}\times\{0\}} \to k_{\Delta_E\times\{0\}}$, which induces from $\Delta_{(-1,1)} \times \{0\} \subset \Delta_E \times \{0\}$.

2. (Microlocal cut-off) Let

$$
\begin{aligned}
Z^+ &= \{(x_2, x_1, t_2, t_1) : |x_2 - x_1| \leq t_2 - t_1\}, \\
Z &= \{(x_2, x_1, t) : |x_2 - x_1| \leq t\}.
\end{aligned}
$$

Then $Z^+ = (m_{32}^{\mathbb{R}_t})^{-1} Z = (m_{32}^{V})^{-1}\gamma_1, k_{Z^+} = (m_{32}^{\mathbb{R}_t})^{-1} k_Z = (m_{32}^{V})^{-1} k_{\gamma_1}$.

Moreover,

$$
\begin{aligned}
k_Z \overset{t}{\star} F &= Rq_{31!}((m_{32}^{\mathbb{R}_t})^{-1} k_Z \otimes q_{21}^{-1} F) \\
&= Rq_{31!}(k_{Z^+} \otimes q_{21}^{-1} F) \\
&= Rq_{31!}((m_{32}^{V})^{-1} k_{\gamma_1} \otimes q_{21}^{-1} F) \\
&= k_{\gamma_1} \overset{V}{\star} F.
\end{aligned}
$$

Also, $k_\gamma \overset{V}{\star} F \to F$ is given by $k_Z \to k_{\Delta_E\times\{0\}}$, which is induces from $Z \supset \Delta_E \times \{0\}$.

1. It follows 1 and 2, that $K_0 = k_Z \underset{x_2}{\overset{t}{\star}} k_{\Delta_{(-1,1)}\times\{0\}}$. We need to check that $k_Z \underset{x_2}{\overset{t}{\star}} k_{\Delta_{(-1,1)}\times\{0\}} \cong k_{W_0}$.

$$\begin{array}{ccccc} & & E_3 \times E_2 \times E_1 \times \mathbb{R}_{t_2} \times \mathbb{R}_{t_1} & & \\ & \swarrow^{m_{32}} & \downarrow{\scriptstyle q_{31}} & \searrow^{q_{21}} & \\ E_3 \times E_2 \times \mathbb{R}_{t_2} & & E_3 \times E_1 \times \mathbb{R}_{t_2} & & E_2 \times E_1 \times \mathbb{R}_{t_1} \end{array}$$

In fact,

$$\begin{aligned} & k_Z \underset{x_2}{\overset{t}{\star}} k_{\Delta_{(-1,1)}\times\{0\}} \\ =& Rq_{31!}\left(m_{32}^{-1}k_Z \otimes q_{21}^{-1}k_{\Delta_{(-1,1)}\times\{0\}}\right) \\ =& Rq_{31!}\left(k_{m_{32}^{-1}Z\cap q_{21}^{-1}(\Delta_{(-1,1)}\times\{0\})}\right), \end{aligned}$$

where

$$\begin{aligned} & m_{32}^{-1}Z \cap q_{21}^{-1}\left(\Delta_{(-1,1)} \times \{0\}\right) \\ =& \{(x_3, x_2, x_1, t_2, t_1) : |x_3 - x_2| \leq t_2 - t_1, t_1 = 0, x_2 = x_1, |x_1| < 1\}. \end{aligned}$$

Noticed $q_{31} : m_{32}^{-1}Z \cap q_{21}^{-1}\Delta_{(-1,1)} \times \{0\} \to W_0$ is a bijection.

Then $K_0 = k_Z \underset{x_2}{\overset{t}{\star}} k_{\Delta_{(-1,1)}\times\{0\}} = k_{W_0}$.

One can check that the diagram is commutative

$$\begin{array}{ccccc} k_{W_0}\overset{t}{\star}F & \xrightarrow{\cong} & k_Z\overset{t}{\star}k_{\Delta_{(-1,1)}\times\{0\}}\overset{t}{\star}F & \longrightarrow & k_Z\overset{t}{\star}F \\ \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\ k_{W_0}\overset{t}{\star}F & & \xrightarrow[(k_{W_0}\to k_Z)\overset{t}{\star}F]{} & & k_Z\overset{t}{\star}F. \end{array}$$

So, $G \Rightarrow k_{\gamma_1}\overset{V}{\star}$ is $(k_{W_0} \to k_Z)\overset{t}{\star}$, which is induced by open inclusion $W_0 \subset Z$. Automatically, $G \Rightarrow \mathrm{Id}$ is given by $\phi_{-1} : k_{W_0} \to k_{\Delta_E\times\{0\}}$, which is induced by $W_0 \subset \{(x_2, x_1, t) : |x_2 - x_1| \leq t\} \supset \Delta_E \times \{0\}$.

As a convolution functor, $G = K_0\overset{t}{\star}$ is automatically a functor on $\mathcal{D}(E)$. Moreover, $W \subset \{t \geq 0\}$. Thus if $\mathrm{supp}(F) \subset \{t \geq t_F\}$, there is $\mathrm{supp}(k_W\overset{t}{\star}F) \subset \{t \geq t_F\}$. Therefore, $G(F) = K_0\overset{t}{\star}F$ induces a well-defined convolution functor on $\mathcal{D}^{t+}(E)$.

□

Noticed, one can check $s : V \times V \to V, (v, w) \mapsto v + w$ is proper on $\gamma_1 \times [-1, 1] \times \{t \geq t_F\}$. Hence, $G(F) \cong k_\gamma\overset{V}{\star}_{np}((F)_{(-1,1)\times\mathbb{R}_t})$ for $F \in \mathcal{D}^{t+}(E)$, as a functor.

The microlocal cut-off lemma predicts that $SS(G(F)) \subset V \times \gamma_1^\circ$. However, if we take $F = k_{(x,0)}, |x| < 1$ as a simple example, $G(k_{(x,0)}) \cong k_{(x,0)+\gamma_1}$. Then we could find that if we only apply G one times, we can not control $\mathrm{supp}(G(k_{(x,0)})))$ when t is large enough. But if we apply G twice, say $G^2(k_{(x,0)})$. It still satisfies $SS(G^2(k_{(x,0)})) \subset V \times \gamma_1^\circ$ and moreover $\mathrm{supp}(G^2(k_{(x,0)}))) \cap \{t \leq 2\} \subset \{|x| \leq 1\}$.

So we could apply G infinitely many times, to make sure we can always control both support and microsupport. The problem here is what is a infinite composition of a functor? In general, this does not make sense. Luckily, we have represented G as a convolution kernel. We could glue directly kernels of G^n, to obtain a sheaf as a convolution kernel, which is a formulation of "G^∞" That is what we are going to do here. Now, let us compute kernel of G^n first.

Proposition 4.2. *Let* $G^{n+1}(F) = K_n \overset{t}{\star} F$, *where* $K_n = \underbrace{K_0 \overset{t}{\star} K_0 \overset{t}{\star} \cdots \overset{t}{\star} K_0}_{n+1 \quad times}$.

Then K_n *is a bounded complex of sheaves:*

$$0 \longrightarrow k_{W_0} \xrightarrow{d^0} \begin{matrix} k_{W_1^+} \\ \oplus \\ k_{W_1^-} \end{matrix} \xrightarrow{d^1} \begin{matrix} k_{W_2^+} \\ \oplus \\ k_{W_2^-} \end{matrix} \xrightarrow{d^2} \cdots \xrightarrow{d^n} \begin{matrix} k_{W_n^+} \\ \oplus \\ k_{W_n^-} \end{matrix} \xrightarrow{0} 0,$$

where

$$W_0 = \{(x_2, x_1, t) : |x_2 - x_1| \leq t, |x_1| < 1\},$$
$$W_i^\pm = \{(x_2, x_1, t) : |x_2 \pm (-1)^i| \leq t - (2i - 1 \mp x_1), |x_1| < 1\}, \quad i \geq 1,$$

and

$$d^0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad d^i = \begin{pmatrix} 1 & (-1)^{i-1} \\ (-1)^{i-1} & 1 \end{pmatrix}, \quad 1 \leq i \leq n-1,$$
$$d^i = 0, \quad others.$$

Here 1 are natural morphisms induced by closed inclusions.

The natural morphism $\phi_n = K_0 \overset{t}{\star} \phi_{n-1} : K_{n+1} \to K_n$ *is given by*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & k_{W_0} & \xrightarrow{d^0} & \begin{matrix} k_{W_1^+} \\ \oplus \\ k_{W_1^-} \end{matrix} & \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} & \begin{matrix} k_{W_n^+} \\ \oplus \\ k_{W_n^-} \end{matrix} & \xrightarrow{d^n} & \begin{matrix} k_{W_{n+1}^+} \\ \oplus \\ k_{W_{n+1}^-} \end{matrix} & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow & & \\ 0 & \longrightarrow & k_{W_0} & \xrightarrow{d^0} & \begin{matrix} k_{W_1^+} \\ \oplus \\ k_{W_1^-} \end{matrix} & \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} & \begin{matrix} k_{W_n^+} \\ \oplus \\ k_{W_n^-} \end{matrix} & \xrightarrow{d^n} & 0 & \longrightarrow & 0. \end{array}$$

Proof. Let us do induction,

$$
\begin{array}{ccccc}
 & & E_3 \times E_2 \times E_1 \times \mathbb{R}_{t_2} \times \mathbb{R}_{t_1} & & \\
 & \swarrow^{m_{32}} & \downarrow^{q_{31}} & \searrow^{q_{21}} & \\
E_3 \times E_2 \times \mathbb{R}_{t_2} & & E_3 \times E_1 \times \mathbb{R}_{t_2} & & E_2 \times E_1 \times \mathbb{R}_{t_2}
\end{array}
$$

1. We compute K_n first.

When $n = 0$, we have known $K_0 = k_{W_0}$ in proposition 4.1. Next, let us compute $K_{n+1} = K_0 \overset{t}{\star} K_n$, which is the image of the following complex, under $Rq_{31!}$.

$$
m_{32}^{-1}(K_0) \otimes q_{21}^{-1} K_n : 0 \longrightarrow k_{A_0} \xrightarrow{d^0} \begin{matrix} k_{A_1^+} \\ \oplus \\ k_{A_1^-} \end{matrix} \xrightarrow{d^1} \cdots \xrightarrow{d^{n-2}} \begin{matrix} k_{A_{n-1}^+} \\ \oplus \\ k_{A_{n-1}^-} \end{matrix} \xrightarrow{d^{n-1}} \begin{matrix} k_{A_n^+} \\ \oplus \\ k_{A_n^-} \end{matrix} \longrightarrow 0,
$$

where

$$
A_0 \coloneqq m_{32}^{-1} W_0 \cap q_{21}^{-1} W_0, \quad A_i^{\pm} \coloneqq m_{32}^{-1} W_0 \cap q_{21}^{-1} W_i^{\pm}, \quad i \geq 1.
$$

Unfortunately, the complex $X_{n+1} \coloneqq m_{32}^{-1}(K_0) \otimes q_{21}^{-1} K_n$ is not $q_{31!}$-acyclic, so we need a $q_{31!}$-acyclic resolution of X_{n+1} to compute the derived functor.

Claim: the following morphism of complexes λ is a quasi-isomorphism.

$$
\begin{array}{cccccccccccccc}
X_{n+1}: & 0 & \longrightarrow & k_{A_0} & \xrightarrow{d^0} & \begin{matrix} k_{A_1^+} \\ \oplus \\ k_{A_1^-} \end{matrix} & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & \begin{matrix} k_{A_n^+} \\ \oplus \\ k_{A_n^-} \end{matrix} & \xrightarrow{0} & 0 & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \lambda_{n+1}} & & & \Big\downarrow & & \Big\downarrow & & & & \Big\downarrow & & \Big\downarrow & & \\
R_{n+1}: & 0 & \longrightarrow & k_{\widetilde{A_0}} & \xrightarrow{d^0} & \begin{matrix} k_{\widetilde{A_1^+}} \\ \oplus \\ k_{\widetilde{A_1^-}} \end{matrix} & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & \begin{matrix} k_{\widetilde{A_n^+}} \\ \oplus \\ k_{\widetilde{A_n^-}} \end{matrix} & \xrightarrow{d^n} & \begin{matrix} k_{B_n^+} \\ \oplus \\ k_{B_n^-} \end{matrix} & \longrightarrow & 0,
\end{array}
$$

where

$$
\begin{aligned}
\widetilde{A_0} &\coloneqq m_{32}^{-1} \overline{W_0} \cap q_{21}^{-1} W_0, \\
&= \{(x_3, x_2, x_1, t_2, t_1) : |x_3 - x_2| \leq t_2 - t_1,\ |x_1| < 1 \\
&\qquad |x_2| \leq 1; |x_2 - x_1| \leq t_1\}, \\
\widetilde{A_i^{\pm}} &\coloneqq m_{32}^{-1} \overline{W_0} \cap q_{21}^{-1} W_i^{\pm}, \\
&= \{(x_3, x_2, x_1, t_2, t_1) : |x_3 - x_2| \leq t_2 - t_1,\ |x_1| < 1 \\
&\qquad |x_2| \leq 1; |x_2 \pm (-1)^i| \leq t_1 - (2i - 1 \mp x_1)\}, \\
B_i^{\pm} &\coloneqq \{(x_3, x_2, x_1, t_2, t_1) : |x_3 - x_2| \leq t_2 - t_1,\ |x_1| < 1 \\
&\qquad x_2 = \pm(-1)^i; |x_2 \pm (-1)^i| \leq t_1 - (2i - 1 \mp x_1)\} \\
&= \{(x_3, x_2, x_1, t_2, t_1) : |x_3 - x_2| \leq t_2 - t_1,\ |x_1| < 1 \\
&\qquad x_2 = \pm(-1)^i; 0 \leq t_1 - (2i + 1 \mp x_1)\},
\end{aligned}
$$

and λ_{n+1} are induced by open inclusions $A_0 \subset \widetilde{A_0}$, $A_i^{\pm} \subset \widetilde{A_i^{\pm}}$, d^n is induced by closed inclusions $B_{i-1}^{\pm} \subset \widetilde{A_i^{\pm}}$.

In fact, when restricted on the domain $\{|x_2| < 1\}$, λ_{n+1} is the identity morphism. When restricted on $\{|x_2| = \pm 1\}$, X_{n+1} is the zero complex, so we only need to check the second row is a exact complex. Stackwisely, the complex is:

$$0 \longrightarrow k \xrightarrow{d^0} k^2 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} k^2 \xrightarrow{d^n} k \longrightarrow 0$$

where

$$d^0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad d^i = \begin{pmatrix} 1 & (-1)^{i-1} \\ (-1)^{i-1} & 1 \end{pmatrix}, \quad n-1 \geq i \geq 1,$$

and $d^n = (-1)^{n-1}\begin{pmatrix} 1 & (-1)^{n-1} \end{pmatrix}$, all 1 here are the $1 \in k$. So it is exact.

A Direct computation shows $q_{31}|_{A_i^{\pm}}$ and $q_{31}|_{B_i^{\pm}}$ are proper with contractible fibre. So the R_{n+1} is a $q_{31!}$-acyclic resolution of X_{n+1}. Therefore,

$$K_{n+1} = Rq_{31!}(X_{n+1}) = q_{31!}(R_{n+1})$$

So the result follows from

$$q_{31}(\widetilde{A_0}) = W_0, \quad q_{31}(\widetilde{A_i^{\pm}}) = W_i^{\pm}, \qquad q_{31}(\widetilde{B_i^{\pm}}) = W_i^{\pm},\ i \geq 1.$$

2. Noticed, the resolution above is compatible with the induction process. More precisely, we have a commutative diagram:

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{\lambda_{n+1}} & R_{n+1} \\ {\scriptstyle \mathrm{id}_{m_{32}^{-1}K_0} \otimes q_{21}^{-1}\phi_{n-1}} \downarrow & & \downarrow {\scriptstyle \tilde{\phi}_n} \\ X_n & \xrightarrow{\lambda_n} & R_n, \end{array}$$

where

$$\tilde{\phi}_n^i = \begin{cases} \mathrm{id} : k_{\widetilde{A_i^+}} \oplus k_{\widetilde{A_i^-}} \to k_{\widetilde{A_i^+}} \oplus k_{\widetilde{A_i^-}}, & 0 \leq i \leq n-1, \\ k_{\widetilde{A_n^+}} \oplus k_{\widetilde{A_n^-}} \to k_{B_{n-1}^+} \oplus k_{B_{n-1}^-}, & i = n, \\ 0, & \text{others}, \end{cases}$$

are induced by closed inclusions.

Therefore,

$$\phi_n = K_0 \overset{t}{\star}(\phi_{n-1}) = Rq_{31!}(\mathrm{id}_{m_{32}^{-1}K_0} \otimes q_{21}^{-1}\phi_{n-1}) = q_{31!}(\tilde{\phi}_n).$$

By definition of $\tilde{\phi}_n$, we can see the ϕ_n is exactly what want.

$\square$

Corollary 4.3. *K_n as above, its cohomology sheaves are:*

$$\mathcal{H}^i(K_n) \cong \begin{cases} k_{P_i}, & 0 \le i \le n-1, \\ k_{W_n}, & i = n, \\ 0, & \textit{others}, \end{cases}$$

where

$$\begin{aligned} P_i &= \{(x_2, x_1, t) : |x_2 - (-1)^i x_1| \le t - 2i, |x_2 + (-1)^i x_1| < 2i + 2 - t, |x_1| < 1\}, \\ W_i &= \{(x_2, x_1, t) : |x_2 - (-1)^i x_1| \le t - 2i, |x_1| < 1\}, \quad i \ge 0. \end{aligned}$$

Definition 4.4. Define K_∞ as a complex of sheaves as follow

$$0 \longrightarrow k_{W_0} \xrightarrow{d^0} \begin{matrix} k_{W_1^+} \\ \oplus \\ k_{W_1^-} \end{matrix} \xrightarrow{d^1} \cdots \xrightarrow{d^{i-1}} \begin{matrix} k_{W_i^+} \\ \oplus \\ k_{W_i^-} \end{matrix} \xrightarrow{d^i} \cdots,$$

where

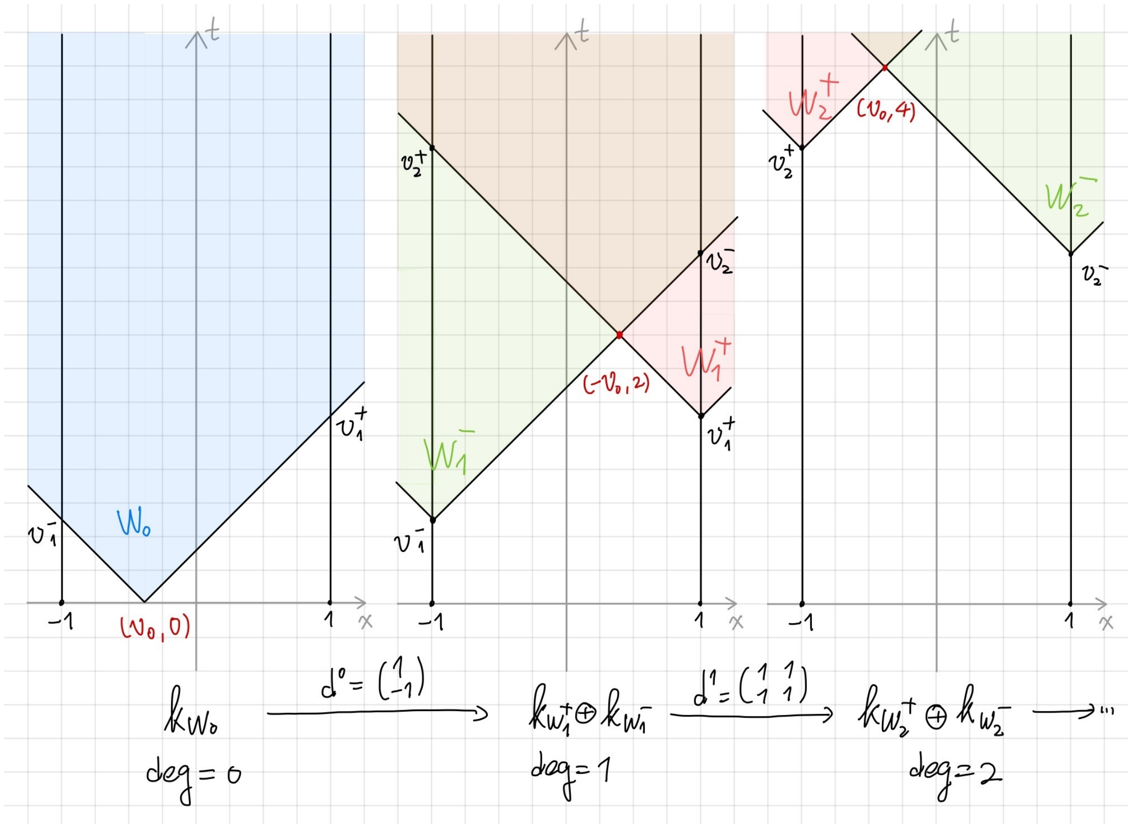
$$d^0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad d^i = \begin{pmatrix} 1 & (-1)^{i-1} \\ (-1)^{i-1} & 1 \end{pmatrix}, \quad i \ge 1,$$

and a morphism Φ_n as follow

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & k_{W_0} & \xrightarrow{d^0} & \begin{matrix} k_{W_1^+} \\ \oplus \\ k_{W_1^-} \end{matrix} & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & \begin{matrix} k_{W_n^+} \\ \oplus \\ k_{W_n^-} \end{matrix} & \xrightarrow{d^n} & \begin{matrix} k_{W_{n+1}^+} \\ \oplus \\ k_{W_{n+1}^-} \end{matrix} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & k_{W_0} & \xrightarrow{d^0} & \begin{matrix} k_{W_1^+} \\ \oplus \\ k_{W_1^-} \end{matrix} & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & \begin{matrix} k_{W_n^+} \\ \oplus \\ k_{W_n^-} \end{matrix} & \xrightarrow{d^n} & 0 & \longrightarrow & 0 \end{array}$$

We have a commutative diagram, for $n \ge m$,

$$K_\infty \xrightarrow{\Phi_n} K_n \xrightarrow{\phi_{n-1} \circ \cdots \circ \phi_m} K_m \longrightarrow k_{\Delta \times \{0\}}, \qquad K_\infty \xrightarrow{\Phi_m} K_m.$$



The slice $K_\infty|_{E_2 \times \{v_0\} \times \mathbb{R}_t}$

Using the notation of corollary 4.3, we could see $\mathcal{H}^i(K_\infty) \cong k_{P_i}, i \geq 0$. Let $U_n = \{(x_2, x_1, t) : t < 2n\}$. Because Φ_n are induced by closed inclusions, one has $\Phi_n|_{U_n} : K_\infty|_{U_n} \xrightarrow{\cong} K_n|_{U_n}$.

Taking cone of Φ_n, say K_n', we obtain a distinguished triangle

$$K_\infty \xrightarrow{\Phi_n} K_n \longrightarrow K_n' \xrightarrow{+1} .$$

The isomorphism $\Phi_n|_{U_n}$ implies $\mathrm{supp}(K_n') \subset U_n^c = \{t \geq 2n\}$.

We will use the distinguished triangle to show K_∞ is in fact the square projector we want.

Theorem 4.5. *Let $F \in \mathcal{D}^{t+}(E)$. Then the following are all equivalent.*

a) the natural morphism $K_\infty \overset{t}{\star} F \to F$ is an isomorphism.

b) $SS(F) \subset \widehat{\square}$

c) $SS(F) \subset \widehat{\square}$ and $F|_{\{\pm 1\} \times \mathbb{R}_t} \cong 0$

Proof. $F \in \mathcal{D}^{t+}(E)$, we may assume $\mathrm{supp}(F) \subset \{t \geq t_F\}$.

Taking convolution with F provides us a distinguished triangle for every $n \in \mathbb{Z}_{\geq 0}$

$$K_\infty \overset{t}{\star} F \xrightarrow{\Phi_n \overset{t}{\star} F} K_n \overset{t}{\star} F \longrightarrow K'_n \overset{t}{\star} F \xrightarrow{+1} .$$

a) $\Rightarrow$ b) If $K_\infty \overset{t}{\star} F \cong F$, restrict above distinguished triangle on $U_\lambda = \{t < \lambda\}$.

$$K_\infty \overset{t}{\star} F|_{U_\lambda} \xrightarrow{\Phi_n \overset{t}{\star} F} K_n \overset{t}{\star} F|_{U_\lambda} \longrightarrow K'_n \overset{t}{\star} F|_{U_\lambda} \xrightarrow{+1} .$$

If $\lambda < 2n + t_F$, there is an isomorphism

$$F|_{U_\lambda} \cong K_\infty \overset{t}{\star} F|_{U_\lambda} \xrightarrow{\cong} K_n \overset{t}{\star} F|_{U_\lambda}.$$

Therefore, by the fact $G(F) \cong k_\gamma \overset{V}{\star}_{np}((F)_{(-1,1)\times \mathbb{R}_t})$, the microlocal cut-off lemma shows us

$$SS(K_\infty \overset{t}{\star} F|_{U_\lambda}) = SS(K_n \overset{t}{\star} F|_{U_\lambda}) \subset SS(G(G^n(F))) \subset E \times \mathbb{R}_t \times \gamma_1^\circ$$

For arbitrary n, η, which implies that $SS(K_\infty \overset{t}{\star} F) \subset E \times \mathbb{R}_t \times \gamma_1^\circ$.

Because $\mathcal{H}^i(K_\infty) \cong k_{P_i}$, $\mathrm{supp}(\mathcal{H}^i(K_\infty)) \subset \overline{P_i} \subset [-1,1] \times [-1,1] \times [2i, 2i+2]$. Then $\mathrm{supp}(K_\infty) = \overline{\cup_i \mathrm{supp}(\mathcal{H}^i(K_\infty))} \subset [-1,1] \times [-1,1] \times \mathbb{R}_t$. Subsequently, $\mathrm{supp}(F) = \mathrm{supp}(K_\infty \overset{t}{\star} F) \subset [-1,1] \times \mathbb{R}_t$.

So the result follows from the fact that $\pi(SS(F)) = \mathrm{supp}(F)$, where π is the cotangent bundle projection.

b) $\Rightarrow$ c) Only need to show $F|_{\{\pm 1\}\times \mathbb{R}_t} \cong 0$. Consider the distinguished triangle:

$$R\Gamma_Z(F) \to F \to R\Gamma_{V\setminus Z}(F) \xrightarrow{+1} .$$

Let $f_\pm(x,t) = 1 \mp x$, $Z_\pm = \{f_\pm \geq 0\}$. We have

$$Z_\pm \supset [-1,1] \times \mathbb{R}_t \supset \mathrm{supp}(F).$$

So the distinguished triangle provides us an isomorphism

$$R\Gamma_{\{f_\pm \geq 0\}}(F) \xrightarrow{\cong} F.$$

Moreover, we also have

$$f_\pm(\pm 1, t) = 0, \quad (df_\pm)_{(\pm 1, t)} = (\pm 1, t, \mp 1, 0) \notin SS(F).$$

So by definition of microsupport, and the above isomorphism, there is

$$0 \cong \left(R\Gamma_{\{f_\pm \geq 0\}}(F)\right)_{(\pm 1, t)} \xrightarrow{\cong} F_{(\pm 1, t)}.$$

It is exactly $F|_{\{\pm 1\} \times \mathbb{R}_t} \cong 0$.

c) $\Rightarrow$ a) If $SS(F) \subset \widehat{\square}$ and $F|_{\{\pm 1\} \times \mathbb{R}_t} \cong 0$.

Then $(F)_{\{|x| \geq 1\} \times \mathbb{R}_t} \cong 0$, because $\pi(SS(F)) = \mathrm{supp}(F)$.

Excision triangle

$$(F)_{(-1,1) \times \mathbb{R}_t} \to F \to (F)_{\{|x| \geq 1\} \times \mathbb{R}_t} \xrightarrow{+1},$$

implies

$$(F)_{(-1,1) \times \mathbb{R}_t} \xrightarrow{\cong} F.$$

On the other hand, $SS(F) \subset \widehat{\square} \subset E \times \mathbb{R}_t \times \gamma_1^\circ$. Then microlocal cut-off lemma provides us an isomorphism:

$$k_\gamma \overset{V}{\star}_{np}((F)_{(-1,1) \times \mathbb{R}_t}) \xrightarrow{\cong} k_\gamma \overset{V}{\star}((F)_{(-1,1) \times \mathbb{R}_t}) \xrightarrow{\cong} k_\gamma \overset{V}{\star} F \xrightarrow{\cong} F.$$

In the point of view of proposition 4.1, this means exactly, $K_0 \overset{t}{\star} F \xrightarrow{\cong} F$.

Inductively, there is $K_n \overset{t}{\star} F \xrightarrow{\cong} F$. Combine with the commutative diagram

$$\begin{array}{ccccc} K_\infty & \longrightarrow & K_n & \longrightarrow & k_{\Delta \times \{0\}}, \\ & \searrow & & \nearrow & \end{array}$$

We obtain, by TR4, a morphism of distinguished triangles

$$\begin{array}{ccccccc} K_\infty \overset{t}{\star} F & \xrightarrow{\Phi_n \overset{t}{\star} F} & K_n \overset{t}{\star} F & \longrightarrow & K'_n \overset{t}{\star} F & \xrightarrow{+1} & \\ \downarrow \mathrm{id} & & \downarrow \cong & & \downarrow & & \\ K_\infty \overset{t}{\star} F & \longrightarrow & F & \longrightarrow & C & \xrightarrow{+1} & , \end{array}$$

where C is a cone of $K_\infty \overset{t}{\star} F \to F$.

Because of the first two isomorphisms, the dotted arrow $K'_n \overset{t}{\star} F \to C$ is an isomorphism for any n.

However, $\mathrm{supp}(K'_n \overset{t}{\star} F) \subset \{t \geq t_F + 2n\}$ while $\mathrm{supp}(C)$ is independent with n. Hence the equality of support holds for any n, and it must have $\mathrm{supp}(C) = \varnothing$. This is exactly means

$$K_\infty \overset{t}{\star} F \to F,$$

is an isomorphism.

$\square$

Corollary 4.6. *The functor*

$$K_\infty \overset{t}{\star} : \mathcal{D}^{t+}(E) \to \mathcal{D}^{t+}(E), F \mapsto K_\infty \overset{t}{\star} F,$$

is a projector, whose essential image is the full subcategories of $\mathcal{D}^{t+}(E)$ *given by the conditions* $SS(F) \subset \widehat{\square}$. *We call both* K_∞ *and the functor* $K_\infty \overset{t}{\star}$ *a* ***square projector****.*

4.2 Displacement Energy of Square projector

Theorem 4.7. *If* $\mathrm{char}(k) \neq 2$, $e'(K_\infty) \leq e(K_\infty) \leq 4$.

Proof. The first inequality is corollary of proposition 3.31. We only need to prove the second inequality. K_∞ is the complex:

$$0 \longrightarrow k_{W_0} \xrightarrow{d^0} \begin{matrix} k_{W_1^+} \\ \oplus \\ k_{W_1^-} \end{matrix} \xrightarrow{d^1} \begin{matrix} k_{W_2^+} \\ \oplus \\ k_{W_2^-} \end{matrix} \xrightarrow{d^2} \begin{matrix} k_{W_3^+} \\ \oplus \\ k_{W_3^-} \end{matrix} \xrightarrow{d^3} \cdots \xrightarrow{d^n} \cdots,$$

where

$$d^0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad d^2 = \begin{pmatrix} 1 & (-1)^{i-1} \\ (-1)^{i-1} & 1 \end{pmatrix}, \quad i \geq 0,$$

and all 1 here are morphisms of sheaves induced by closed inclusions.

By example 3.28, one can see $\tau_c : K_\infty \to T_{c*}K_\infty$ are induced by close inclusions $W_i^\pm \supset T_c(W_i^\pm)$. Using matrices, there is

$$\tau_c^0 = 1, \quad \tau_c^i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \geq 1.$$

Here, we shall construct a chain homotopy between $\tau_c, c \geq 4$ and the zero map of K_∞ to show $e(K_\infty) \leq 4$.

In fact, if $c \geq 4$, there are some close inclusions:

$$T_c(W_0) \subset W_1^\pm,$$
$$T_c(W_i^\pm) \subset W_{i+1}^\pm, \quad i \geq 1.$$

These close inclusion induces a map of degree -1, say $H : K_\infty \to K_\infty[-1]$, as follow

$$H^0 = 0, \quad H^1 = \frac{1}{2}\begin{pmatrix} 1 & -1 \end{pmatrix}, \quad H^i = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \geq 2.$$

Now, let us check H is indeed a chain homotopy between τ_c and 0. Noticed that composition of closed inclusions are still closed inclusions. Reflected in the calculation, it means $1 \times 1 = 1$ when doing formal computation of matrices of morphisms. So

$$
\begin{aligned}
&d^{-1}H_0 + H^1 d^0 = 0 + \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1 \\ -1\end{pmatrix} = 1,\\
&d^0 H^1 + H^2 d^1 = \begin{pmatrix}1 \\ -1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 \\ 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix},\\
&d^i H^{i+1} + H^{i+2} d^{i+1} = \frac{1}{2}\begin{pmatrix}1 & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -1 \\ -1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}, \ i \geq 1.
\end{aligned}
$$

It is directly to see all results of these equalities are τ_c^i, i.e. $\tau_c = dH + Hd$, which exactly means τ_c chain homotopy to zero. In particular, it means τ_c is zero as a morphism in derived categories.

Then $e(K_\infty) \leq 4$. □

as a corollary of proposition 3.31, proposition 3.33, theorem 4.5.

Corollary 4.8. *Let* $F_{x_0} = K_\infty \overset{t}{\star}_{(x_0,0)} \cong K_\infty|_{E \times \{x_0\} \times \mathbb{R}_t}$, $x_0 \in (-1,1)$. *Then* $e'(F_{x_0}) \leq e(F_{x_0}) \leq 4$.

4.3 Cube and Square Prism Let us fix some conventions in this section.

1. $[d] \coloneqq \{1, 2, \cdots, d\}$.

2. Vectors are in boldface $\mathbf{x} = (x_1, ...x_d) \in E$. And if there are two vector spaces E_1 and E_2, we will use $\mathbf{x}_1 = (x_{11}, ..., x_{1d}) \in E_1, \mathbf{x}_2 = (x_{21}, ...x_{2d}) \in E_2$. Similar conventions holds for dual spaces E^*. $\alpha \in [d]$

3. ∞ only means positive infinity. If $u, v \in \mathbb{R}$, u/∞ means 0, and $|u| \leq \infty v$ means $|u| < \infty$.

In this section, We will constructurt kernels associate to higher dimensional cubes based on the square case.

For a rectangle, we mean

$$\square^2(s, l) = \{(x, p) : |x| \leq s/2, |p| \leq l/2\},$$

Here, $s, l \in (0, \infty]$, and they are not infinity at the same time. In dimension two, symplectic geometry is exactly area-preserving geometry. So the most important invaraints are the area of $\square^2(s, l)$, say sl. Two rectangles are symplectomorphic as symplectic manifolds (as symplectic manifolds with corner) if and only they have the same area.

In higher dimension, we call a cube as product of rectangles, say

$$C^d(\mathbb{s}) \coloneqq \square^2(s_1, l_1) \times \cdots \times \square^2(s_d, l_d),$$

where $\mathbb{s} = (l, s_1, s_2, \cdots, s_d, I(\mathbb{s})) \in (0, \infty]^{d+1} \times 2^{[d]}$, $I(\mathbb{s}) = \{\alpha \in [d] : l_\alpha = \infty\}$.

As remarkd before, in higher dimension, we assume all non-infinity l_α are equal to l and $s_\alpha = \infty$ if and only $\alpha \in I(\mathbb{s})$.

For example, $\square$ in §4.1 is $C^1(2, 2, \varnothing)$, and the total space is $C^d(\infty, \cdots, \infty, [d])$.

We call $P_l^d = C^d(l, l, \infty, \cdots, \infty, [d] \setminus \{1\})$ a regular prism, and we also call $\square_l^d = C^d(l, \cdots, l, \varnothing)$ a regular cube.

For the usage to microsupport, we also need their cone lifting, say

$$\widehat{C^d}(\mathbb{s}) = \overline{\rho^{-1}(C^d(\mathbb{s}))}, \widehat{\square}_l^d = \overline{\rho^{-1}(\square_l^d)}, \widehat{P}_l^d = \overline{\rho^{-1}(P_l^d)}.$$

Let $\gamma_{\mathbb{s}} = \{(\mathbf{x}, t) : |x_\alpha| \leq 2t/l_\alpha, \forall \alpha\}$ be a close cone in V. Its polarized cone is $\gamma_{\mathbb{s}}^\circ = \{(\mathbf{p}, \tau) : |p_\alpha| \leq l_\alpha \tau/2, \forall \alpha\}$.

So, one can consider functors

$$G(\mathbb{s})(F) : \mathcal{D}^{t+}(E) \to \mathcal{D}^{t+}(E), \quad k_{\gamma_{\mathbb{s}}} \overset{V}{\star} (F_{\prod_{\alpha=1}^d (-s_\alpha, s_\alpha) \times \mathbb{R}_t})$$

For example, $G = G((2, 2, \varnothing))$.

Proposition 4.9. *$G(\mathbb{s})(F) = K_0(\mathbb{s}) \overset{t}{\star} F$, where $K_0(\mathbb{s}) = k_{W_0(\mathbb{s})}$, and*

$$W_0(\mathbb{s}) = \{(\boldsymbol{x}_2, \boldsymbol{x}_1, t) \in E_2 \times E_1 \times \mathbb{R}_t : |x_{2\alpha} - x_{1\alpha}| \leq 2t/l_\alpha, |x_{1\alpha}| < s_\alpha/2, \forall \alpha\}.$$

The morphism of functor $G(\mathbb{s}) \Rightarrow Id$ is given by $\phi_{-1} : k_{W_0(\mathbb{s})} \to k_{\Delta_E \times \{0\}}$, which is induced by $W_0(\mathbb{s}) \subset \{(\boldsymbol{x}_2, \boldsymbol{x}_1, t) : |x_{2\alpha} - x_{1\alpha}| \leq 2t/l_\alpha, \forall \alpha\} \supset \Delta_E \times \{0\}$.

Let $K_n(\mathbb{s}) = K_0(\mathbb{s}) \overset{t}{\star} K_0(\mathbb{s}) \overset{t}{\star} \cdots \overset{t}{\star} K_0(\mathbb{s})$. One could hope that we can construct the cube kernel via a resolution of $K_n(\mathbb{s})$, similar to what we did in the last section. It is possible. However, I will not do such things here, but use directly a convolution construction.

At the beginning, all constructions in the previous section work for prisms cases. Let $\mathbb{p}_{l,\alpha} = (l, \infty, \cdots, \infty, s_\alpha, \infty, \cdots, \infty, [d] \setminus \{i\})$. In the following, we fix l and omit the lower subscript l of $\mathbb{p}_{l,\alpha}$. So we have obtained datas as follow:

1. The kernels $K_n(\mathbb{p}_\alpha) = K_0(\mathbb{p}_\alpha)\overset{t}{\star}K_0(\mathbb{p}_\alpha)\overset{t}{\star}\cdots\overset{t}{\star}K_0(\mathbb{p}_\alpha)$ and $K(\mathbb{p}_\alpha) \coloneqq K_\infty(\mathbb{p}_\alpha)$.
2. Morphisms: $\Phi(\mathbb{p}_\alpha) : K(\mathbb{p}_\alpha) \to K_n(\mathbb{p}_\alpha)$. They induce isomorphisms

   $\Phi(\mathbb{p}_\alpha)|_{\{2t<nls_\alpha\}} : K(\mathbb{p}_\alpha)|_{\{2t<nls_\alpha\}} \xrightarrow{\cong} K_n(\mathbb{p}_\alpha)|_{\{2t<nls_\alpha\}}, \forall\alpha \in [d]$.

Definition 4.10. For a cube of shape index $\mathbb{s}$, we define a kernel as follow:

$$K(\mathbb{s}) \coloneqq K(\mathbb{p}_1)\overset{t}{\star}K(\mathbb{p}_2)\overset{t}{\star}\cdots\overset{t}{\star}K(\mathbb{p}_d)$$

Lemma 4.11. *Using notations above, we have*

a) $K_n(\mathbb{s}) \cong 3K_n(\mathbb{p}_1)\overset{t}{\star}K_n(\mathbb{p}_2)\overset{t}{\star}\cdots\overset{t}{\star}K_n(\mathbb{p}_d)$

b) $K_n(\mathbb{s})|_{2t<nl(s_1+\cdots+s_d)} \cong K(\mathbb{s})|_{2t<nl(s_1+\cdots+s_d)}$

Proof. To be concise, let us assume $d = 2$. It is direct to extend the proof for all d.

a) Noticed that $K_0(\mathbb{s}) \cong K_0(\mathbb{p}_1)\overset{t}{\star}K_0(\mathbb{p}_2) \cong K_0(\mathbb{p}_2)\overset{t}{\star}K_0(\mathbb{p}_1)$.

   This isomorphism does not use the switch factors map in example 3.13 (3).

   So we can conclude inductively.

b) Consider the following diagram.

$$\begin{array}{ccc}
K(\mathbb{s})|_{\{2t<nl(s_1+s_2)\}} & \longrightarrow & K_n(\mathbb{s})|_{\{2t<nl(s_1+s_2)\}} \\
\Big\downarrow{=} & & \Big\uparrow{\cong} \\
[K(\mathbb{p}_1)\overset{t}{\star}K(\mathbb{p}_2)]|_{\{2t<nl(s_1+s_2)\}} & \longrightarrow & [K_n(\mathbb{p}_1)\overset{t}{\star}K_n(\mathbb{p}_2)]|_{\{2t<nl(s_1+s_2)\}} \\
\Big\downarrow & & \Big\downarrow \\
K(\mathbb{p}_1)|_{\{2t<nls_1\}}\overset{t}{\star}K(\mathbb{p}_2)|_{\{2t<nls_2\}} & \longrightarrow & K_n(\mathbb{p}_1)|_{\{2t<nls_1\}}\overset{t}{\star}K_n(\mathbb{p}_2)|_{\{2t<nls_2\}}
\end{array}$$

   The first horizontal arrow is a composition of three morphisms in upper part of the diagram. The second horizontal arrow is $[\Phi(\mathbb{p}_1)\overset{t}{\star}\Phi(\mathbb{p}_2)]|_{\{2t<nl(s_1+s_2)\}}$. The lower-est horizontal arrow is $\Phi(\mathbb{p}_1)|_{\{2t<nls_1\}}\overset{t}{\star}\Phi(\mathbb{p}_2)|_{\{2t<nls_2\}}$, so it is an isomorphism.

   We need to specify two lower vertical morphisms and check that the diagram is commutative. Therefore, we only need to check two vertical arrows in the lower part are isomorphisms.

   In fact, we have

$$\begin{array}{ccc}
[K(\mathbb{p}_1)\overset{t}{\star}K(\mathbb{p}_2)]|_{\{2t<nl(s_1+s_2)\}} & \longleftarrow & K(\mathbb{p}_1)|_{\{2t<nls_1\}}\overset{t}{\star}K(\mathbb{p}_2)|_{\{2t<nls_2\}} \\
\downarrow{=} & & \downarrow{=} \\
Rs_!([q_{32}^{-1}K_1 \otimes q_{21}^{-1}K_2]|_{\{2(t_1+t_2)<nl(s_1+s_2)\}}) & \longleftarrow & Rs_!([q_{32}^{-1}K_1 \otimes q_{21}^{-1}K_2]|_{\{2t_1<nls_1, 2t_2<nls_2\}})
\end{array}$$

The second row is induces from open inclusion

$$\{2t_1 < nls_1, 2t_2 < nls_2)\} \subset \{2(t_1 + t_2) < nl(s_1 + s_2)\},$$

so we can take the first row as what we want.

Moreover, $\mathrm{supp}(K_i) \subset \{t_i \geq 0\}$. So s is proper with contractible fibre.

Hence, the isomorphism comes from

$$s(\{2t_1 < nls_1, 2t_2 < nls_2)\}) = s(\{2(t_1 + t_2) < nl(s_1 + s_2)\}).$$

Similarly, the right vertical map is also an isomorphism.

□

Based on the part *b*) of the lemma, we can conclude the following theorem using same arguement with theorem 4.5.

Theorem 4.12. *Let $F \in \mathcal{D}^{t_+}(E)$. Then the following are all equivalent.*

a) the natural morphism $K(\mathbb{s})\overset{t}{\star}F \to F$ *is an isomorphism.*

b) $SS(F) \subset \widehat{C^d}(\mathbb{s})$.

c) $SS(F) \subset \widehat{C^d}(\mathbb{s})$ *and* $F|_{\{x_i = \pm s_i/2, \forall i, s.t. s_i \neq \infty\} \times \mathbb{R}_t} \cong 0$

Corollary 4.13. *Functor*

$$K(\mathbb{s})\overset{t}{\star} : \mathcal{D}^{t_+}(E) \to \mathcal{D}^{t_+}(E), F \mapsto K(\mathbb{s})\overset{t}{\star}F,$$

*is a projector, whose essential image is the full subcategories of $\mathcal{D}^{t_+}(E)$ given by the conditions $SS(F) \subset \widehat{C^d}(\mathbb{s})$. We call both $K(\mathbb{s})$ and the functor $K(\mathbb{s})\overset{t}{\star}$ a $C^d(\mathbb{s})$-**projector**.*

Theorem 4.14. *If* $\mathrm{char}(k) \neq 2$, *then for any cube $C^d(\mathbb{s})$ excluded the total space, we have $e'(K(\mathbb{s})) \leq e(K(\mathbb{s})) \leq \min\{s_\alpha l : \forall \alpha \notin I(\mathbb{s})\}$.*

For $\boldsymbol{x}_0 \in (-1,1)^n$, let $F_{\boldsymbol{x}_0} = K(\mathbb{s})\overset{t}{\star}k_{(\boldsymbol{x}_0,0)}$, $e'(F_{\boldsymbol{x}_0}) \leq e(F_{\boldsymbol{x}_0}) \leq \min\{s_\alpha l : \forall \alpha \notin I(\mathbb{s})\}$.

4.4 Non-squeezing We are going our final target. In this subsection, we prove the some non-squeezing theorems.

Lemma 4.15. *Let $F \in \mathcal{D}^{t+}(E)$. Suppose $\rho(SS(F))$ is bounded. If there is a symplectomorphism $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\phi(\rho(SS(F))) \subset \mathrm{Int}(P_l^d) = \{(x,p) \in \mathbb{R}^{2n}, |x_1| < l/2, |p| < l/2\}$. Then $e'(F) \leq e(F) \leq l^2$.*

Proof. We can assume $\phi = \phi_1$, where ϕ_s is a compact support Hamiltonian isotopy. In fact, we are working on $\mathbb{R}^{2d}$, so we can assume ϕ_s is Hamiltonian. Besides $\rho(SS(F))$ is bounded, so we can assume ϕ is compact support.

Then ϕ admit a conification $\widehat{\phi_1}$, and we have the GKS sheaf quantization $K(\widehat{\phi_1})$.

By proposition 2.10-b), and properties of GKS sheaf quantization, there is

$$\rho(SS(F \circ K(\widehat{\phi_1}))) = \rho(\phi_1(SS(F))) \subset \mathrm{Int}(P_l^d).$$

Then the projector properties shows us

$$K(P_l^d)\overset{t}{\star}(F \circ K(\widehat{\phi_1})) \cong F \circ K(\widehat{\phi_1}).$$

Then theorem 4.14, proposition 3.33, proposition 3.34 implies

$$4l^2 \geq e(K(P_l^d)) \geq e(F \circ K(\widehat{\phi_1})) = e(F) \geq e'(F).$$

□

Example 4.16. Recall example 3.32. Let $g : [0,\infty) \to [0,\infty)$ be a smooth function. Assume $g'(x) \leq 0$ such that $g(0) > 0$, $g(a) = 0$, and $g'(x) < 0, 0 < x \leq a; g'(x) = 0$ otherwise.

For the smooth function $f : \mathbb{R}^d \to [0,\infty)$, $f(x) = g(\|x\|^2)$.

Let $Z = \{(x,t) \in \mathbb{R}^d \times \mathbb{R}_t :, -f(x) \leq t < f(x)\}$. $k_Z \in \mathcal{D}(\mathbb{R}^d)$, and it fits into the excision distinguished triangle

$$k_Z \to k_{Z_-} \to k_{Z_+} \xrightarrow{+1},$$

where $Z_\pm = \{(x,t) \in \mathbb{R}^d \times \mathbb{R}_t :, t \geq \pm f(x)\}$.

Now let us estimate of k_Z. In fact, by example 3.2,

$$SS(k_{Z_\pm}) = \{(x, \mp f(x), \pm\tau df(x), \tau) : \tau \geq 0\} \cup 0_{Z_\pm}$$

So the triangular inequality shows

$$SS(k_Z) \cap T^*_{\tau>0}(\mathbb{R}^d \times \mathbb{R}_t) \subset SS(k_{Z_+}) \cap SS(k_{Z_-}),$$

and

$$\rho(SS(k_Z)) \subset \{(x, \pm df(x) : \|x\| < a\}.$$

Now if we assume $\rho(SS(k_Z)) \subset B^{2d} = \{\|x\|^2 + \|p\|^2 < 1\}$. Because $f(x) = g(\|x\|^2)$, we have $df(x) = 2g'(\|x\|^2)xdx$. Then

$$\left|\|x\|^2 \left(1 + 4\left(g'(\|x\|^2)\right)^2\right)\right| \leq 1,\ a \leq 1$$

If we take $a = 1$, then we need to solve a differential inequality

$$\begin{cases} t(1 + 4(g'(t))^2) \leq 1 \\ \text{subject to } g(0) > 0,\ g(1) = 0,\ g'(t) < 0,\ 0 < x \leq 1; g'(x) = 0 \text{ otherwise.} \end{cases}$$

Then

$$g'(t) \leq -\frac{1}{2}\sqrt{\frac{1}{t} - 1} \Rightarrow g(t) \leq -\frac{1}{2}\int_1^t \sqrt{\frac{1}{x} - 1}dx$$

Noticed if we can take the integral directly, $g'(0) = \infty$. So we will not take the integral, but taking a smoothing of it near 0 such that $g'(0) = 0$. It not hard to find such a smoothing with $g(0) > 0$. Besides, when $t > 1$, we just extend $g'(t)$ by 0. In this case, g is not C^∞ any more, it is only a C^1 function. But our microsupport estimates still hold. Finally, $g(0) \leq \frac{1}{2}\int_0^1 \sqrt{\frac{1}{x} - 1}dx = \pi/4$. Therefore, result of example 3.32 show us $e(k_Z) = 2g(0) \leq \pi/2$. Moreover, if we take a smoothing near a neighbourhood of 0 which is small enough, we could require $2g(0) = \pi/2 - \varepsilon$. So lemma 4.15 could be used if there is a symplectomorphism $\phi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ such that $\phi(B^{2d}) \subset \mathrm{Int}(P_l^d)$, we have $l^2 \geq \pi/2 - \varepsilon$. In particular, arbitrariness of ε implies that $l \geq \sqrt{\frac{\pi}{2}}$. This is a very weak version of the symplectic non-squeezing.

In this example, one can moreover consider directly that $g(t) = -\frac{1}{2}\int_1^t \sqrt{\frac{1}{x} - 1}dx$. In this case, $\Lambda = \rho(SS(k_Z))$ is not a smooth Lagrangian submanifold. Because $g(t)$ is not differential at 0.

Moreover, one can see that

$$\Lambda \subset B_1(T_0^*\mathbb{R}^d) \cup L,$$

where

$$L = \{\phi_t(S^{d-1}) : S^{d-1} \subset \mathbb{R}^d \subset \mathbb{C}^d\},$$
$$\phi_t(z) = e^{it}z, \quad T^*\mathbb{R}^d \cong \mathbb{C}^d \text{ by } (x,p) \mapsto x + ip.$$

L is a monotone Lagrangian immersion of $\mathbb{C}^d$, which is an embedding if d is even [Pol91].

Guillermou construct a sheaf F in [Gui19], using the k_Z in example 4.16 as a building block, such that $\rho(SS(F)) \subset L$. Moreover, he estimates its displacement energy $e'(F) \geq \pi/2$. So for this L, one can still say it cannot be squeezing into a prism (or a cube) of side length with size l, if $l < \sqrt{\frac{\pi}{2}}$. This is a Lagrangian Non-squeezing result. One can see more on Lagrangian Non-squeezing in [Thé99].

Conjecture 4.17. *$e'(K(\mathbb{s})) \geq \min\{s_\alpha l : \forall \alpha \notin I(\mathbb{s})\}$ for all size index $\mathbb{s}$ except the total space case.*

Some evidence of the conjecture:

1. Directly, one has $e(K(\mathbb{s})) = \min\{s_\alpha l : \forall\alpha \notin I(\mathbb{s})\}$, if this conjecture is ture. So one can use lemma 4.15 to prove: If there is a symplectomorphism $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\phi(\Box_l^d) \subset P_L^d$, then $l \leq L$.

This is a kind of cube Non-squeezing theorem, which is parallel to the Gromov Non-squeezing theorem that states a similar rigidity phenomenons of balls.

2. Let $2d = \min\{s_\alpha l : \forall\alpha \notin I(\mathbb{s})\}$. In fact, one can see easily by theorem 3.5 that

$$\mathrm{Hom}\left(K(\mathbb{s}), T_{c*}(K(\mathbb{s}))\right) \text{ and } \mathrm{Hom}\left(T_{d*}K(\mathbb{s}), T_{(c+d)*}(K(\mathbb{s}))\right), \quad 0 \leq c < d,$$

are constant in the sense that they are stacks of some constant sheaves on $[0,d)$. The only subtlety is what will happen when $V_c = \mathrm{Hom}(K(\mathbb{s}), T_{c*}(K(\mathbb{s})))$ passes $d \in [0, 2d)$. If we can shows it is still constant, then the conjecture is correct.

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