# ERRATA FOR PHD THESIS

(1) The Corollary 2.20 is not true in the generality as it stated.

Thanks to Stéphane Guillermou and Nicolas Vichery for pointing out the mistake in a private communication:

> As it is stated it is certainly false: the open set $U$ is defined by $\{H < 1\}$ with $H : T^*X \to [0,1]$ and there is no special assumption. So we could as well replace $H$ by $H_1 = 1 - (1-H)^2$ which also defines $U$. But the flow of $H_1$ is the identity map on $\partial U$ and the bound reduces to the first part $\{(q,-p,q,p,0) : H(q,p) \leq 1\}$ which clearly not correct. In fact there is probably an issue with the non-characterisitic hypothesis of Theorem 1.4 and we cannot apply the "easy" bounds for the microsupport.

One correction we need would be the following:

**Proposition** (Correction of Corollary 2.20)**.** *Let $U$ be a bounded open set, assume that there exists a smooth autonomous function $H \in C_c^\infty(T^*M) + \mathbb{R}$ (i.e. constant outside of a compact set) such that $U = \{H < 1\}$, $\partial U = \{H = 1\}$ and there exists a relatively compact neighborhood $W$ of $\partial U = \{H = 1\}$ so that, for the standard Liouville form $\lambda$,*

$$\lambda(X_H) > 0 \text{ on } W.$$

*In particular, $1$ is a regular value of $H$ and $\partial U = \{H = 1\}$ is smooth.*

*Then the microlocal kernel $P_U$ satisfies the following microsupport estimate:*

$$\begin{aligned}\mu s_L(P_U) \subset \{(\boldsymbol{q}, -\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{p}, 0) : H(\boldsymbol{q}, \boldsymbol{p}) \leq 1,\ \boldsymbol{q} \in \overline{X_U}\} \\ \cup \{(\boldsymbol{q}, -\boldsymbol{p}, \varphi_z^H(\boldsymbol{q}, \boldsymbol{p}), -\int_c \alpha) : H(\boldsymbol{q}, \boldsymbol{p}) = 1, z < 0,\ \boldsymbol{q} \in \overline{X_U}\},\end{aligned} \tag{1}$$

*Remark* 1.1. Before giving its proof, let make some remarks first. 1) In fact, the condition is equivalent to that $\partial U$ is a restricted contact type (RCT) hypersurface by pick a suitable function $H$. 2) We only use Corollary 2.20 in [Zha22, Lemma 3.28] (i.e. [Zha24, Lemma 2.21]), where RCT condition is stated. So, no more mistakes are maken due to the issue. Moreover, we will see that the way we correct the statement is almost stated in the proof of [Zha22, Lemma 3.28]. 3) It is clear that when $H$ satisfies the condition, $H_1 = 1 - (1-H)^2$ does not.

*Proof.* We refer to notation in the Thesis.

We already have

$$\mu s_L(\widehat{\mathcal{K}}(\widehat{\varphi^H})) \subset \{(H(\mathbf{q}, \mathbf{p}), z, \mathbf{q}, -\mathbf{p}, \varphi_z^H(\mathbf{q}, \mathbf{p}), -S_H(z, \mathbf{q}, \mathbf{p}) - zH(\mathbf{q}, \mathbf{p})) : (z, \mathbf{q}, \mathbf{p}) \in I \times T^*X\}.$$

and

$$SS(\mathbb{K}_{\{\zeta \leq 1\}}) = \{(\zeta, 0) : \zeta \leq 1\} \cup \{(1, z) : z < 0\} \subset \mathbb{R}_z \times \mathbb{R}_\zeta.$$

We shall prove that under the given condition, we have

$$SS(\widehat{\mathcal{K}}(\widehat{\varphi^H})) \cap [SS(\mathbb{K}_{\{\zeta \leq 1\}}) \times 0_{X^2 \times \mathbb{R}_t}] \subset 0_{\mathbb{R}_\zeta \times X^2 \times \mathbb{R}_t}.$$

If this is valid, which enable us to use Equation (1.8), then the rest of the proof is valid.

Since $\widehat{\mathcal{K}}(\widehat{\varphi^H})$ is a Fourier transformation, we use the Tamarkin's non-proper pushforward estimation to derived a contradiction: Assume that there exists $z \neq 0$ such that

$$(1, z, \mathbf{q}, 0, \mathbf{q}', 0, t, 0) \in SS(\widehat{\mathcal{K}}(\widehat{\varphi^H})),$$

then there will be

$$(\mathbf{q}_n, \mathbf{p}_n) \to (\mathbf{q}, 0), \quad z_n, \quad \tau_n > 0$$

such that

$$H(\mathbf{q}_n, \mathbf{p}_n) \to 1, \quad \tau_n \to 0, \quad \tau_n z_n \to z,$$

and

$$S_H(z_n, \mathbf{q}_n, \mathbf{p}_n) - z_n H(\mathbf{q}_n, \mathbf{p}_n)) = -\int_z^0 \lambda(X_H)(\varphi_s^H(\mathbf{q}_n, \mathbf{p}_n) ds \to t.$$

In particular, we have

$$z_n = \tau_n z_n / \tau_n \to -\infty.$$

On the other hand, our positivity condition implies that there exists $\varepsilon, c > 0$ such that

(a) $\{1-\varepsilon \leq H \leq 1+\varepsilon\}$ is compact,
(b) $\lambda(X_H) \geq c$ on $\{1-\varepsilon \leq H \leq 1+\varepsilon\}$.

Therefore
$$|t| \leftarrow |\int_z^0 \lambda(X_H)(\varphi_s^H(\mathbf{q}_n, \mathbf{p}_n))ds| \geq c|z_n|.$$
This is a contradiction since $t$ is a finite number but $|z_n|$ divergent. □

## References

[Zha22] Bingyu Zhang. *Sheaf techniques and symplectic geometry*. Ph.D. thesis, Université Grenoble Alpes [2022]. URL https://byzhang.cn/Files/PhD_Thesis.pdf.

[Zha24] Bingyu Zhang. *Capacities from the Chiu-Tamarkin complex*. Journal of Symplectic Geometry, **volume 22**, no. 3, 441–524 [2024]. doi: 10.4310/jsg.241001211759.