LECTURES ON TORIC VARIETY

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This is the lecture notes for the topics course *Advance topics in Quantum Mathematics* at SDU spring 2025. We will cover basic knowledge on toric variety. The main reference of the note is [CLS11]. We require the readers have a basic knowledge of algebraic geometry (for example, covering [Har77, Section 1.1-1.4, 2.1-2.8], but not necessarily solid).

We will try to achieve two goals: 1) Construction of Calabi-Yau hypersurfaces in toric varieties. 2) Proof of a version of homological mirror symmetry for toric variety (precisely, the so called Coherent-Constructible correspondence).

Throughout, we will consider algebraic varieties over $\mathbb{C}$.

1. Algebraic tori

For an algebraic tori, we mean the variety

$$\mathbb{T}_n = (\mathbb{C}^\times)^n = \{(x_1, \cdots, x_n) : x_i \in \mathbb{C}^\times\}.$$

To emphasize its group structure, we also use the notation $\mathbb{T}_1 = \mathbb{G}_m$ and $\mathbb{T}_n = \mathbb{G}_m^n$, where m here means multiplication. In case there is no confusion, we also write $\mathbb{T}$ without the dimension.

Proposition 1.1. *The variety $\mathbb{T}_n$ is affine and smooth with the coordinate ring $\mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$.*

Proof. We can identify $\mathbb{T}_n \hookrightarrow \mathbb{C}^{2n}$ as

$$\{(x_1, y_1, \cdots, x_n, y_n) : x_i y_n = 1, i = 1, \cdots, n\}.$$

Then it is clear that $\mathbb{T}_n$ is affine and the coordinate ring is identified with

$$\frac{\mathbb{C}[x_1, y_1, \cdots, x_n, y_n]}{\langle x_i y_i - 1; i = 1, \cdots, n = 1\rangle} = \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]. \qquad \square$$

The smoothness can be tested using the defining equation above by the Jacobian criterion.

In fact, the group structure is given by polynomials. So $\mathbb{T}_n = \mathbb{G}_m^n$ is actually an affine algebraic group. It is the simplest reductive algebraic group (we will not use this fact).

Next, we consider characters. We define $M = \mathrm{Hom}_{AlgGrp}(\mathbb{T}_n, \mathbb{G}_m)$ as the multiplicative abelian group of algebraic group homomorphisms (roughly speaking, both polynomial maps and group homomorphisms) induced from multiplicative group structure on $\mathbb{G}_m$.

Proposition 1.2. *We have $M \simeq \mathbb{Z}^n$ (as abelian groups) via the homomorphism*

$$\mathbb{Z}^n \to M, \quad a \mapsto [x \mapsto \chi_a(x) = x_1^{a_1} \cdots x_n^{a_n}].$$

Proof. It is easy to check that the given map is injective group homomorphism. Be careful, we use additive group structure on $\mathbb{Z}^n$ and multiplication on M. It is a group homomorphism means $\chi^{a+b} = \chi_a \chi^b$. Then we only need to check it is surjective.

For $f \in M$, i.e. $f : \mathbb{T}_n \to \mathbb{G}_m$. By Proposition 1.1, and f is a algebraic morphism, we have a ring homomorphism

$$f^* : \mathbb{C}[t^{\pm 1}] \to \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}].$$

Regard $t : \mathbb{G}_m \to \mathbb{G}_m$ as the identity polynomial function. Because t is an invertible element in the ring $\mathbb{C}[t^{\pm 1}]$, we must have $f^*(t)$ invertible in the target. Then we have $f^*(t) = c\chi_a$ for some $c \in \mathbb{C}$. On the other hand, f is a group homomorphism, then we have $f(1, \cdots, 1) = 1$. Then we have

$$c = c\chi_a(1, \cdots, 1) = f^*(t)(1, \cdots, 1) = t(f(1, \cdots, 1)) = t(1) = 1.$$

Then we have $f(x) = t(f(x)) = f^*(t)(x) = \chi_a(x)$, which means that $f = \chi_a$. I.e. all algebraic group homomorphisms $f : \mathbb{T}_n \to \mathbb{G}_m$ are of the form χ_a. $\square$

We have a similar result for cocharacter (or 1-parameter subgroup).

Proposition 1.3. *For the abelian group of algebraic group homomorphism $N = \mathrm{Hom}_{AlgGrp}(\mathbb{G}_m, \mathbb{T}_n)$. We have $N \simeq \mathbb{Z}^n$ (as abelian groups) via the homomorphism*

$$\mathbb{Z}^n \to N, \quad b \mapsto [t \mapsto \lambda_b(t) = (t^{b_1}, \cdots, t^{b_n})].$$

With the identification, we notice that we have an isomorphism

$$\mathbb{T}_n \simeq N \otimes \mathbb{G}_m. \tag{1.1}$$

It is clear we have a pairing between M and N

$$M \times N \to \mathrm{Hom}_{AlgGrp}(\mathbb{G}_m, \mathbb{G}_m), \quad (\chi_a, \lambda_b) \mapsto \chi_a \circ \lambda_b : \mathbb{G}_m \to \mathbb{G}_m.$$

By identification Proposition 1.2 and Proposition 1.3, the pairing matches the perfect pairing

$$\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}, \quad (a, b) \mapsto a_1 b_1 + \cdots a_n b_n.$$

Convention: Normally, we fixed that M (N) to be the (co)character lattice of a tori $\mathbb{T}_n$, as well as their perfect pairing. But we do NOT fix a identification of M, N with $\mathbb{Z}^n$! For an abelian group $\mathbb{K}$, we denote $M_{\mathbb{K}}$, $N_{\mathbb{K}}$ as $M \otimes \mathbb{K}$, $N \otimes \mathbb{K}$. Typically, we take $\mathbb{K} = \mathbb{Q}, \mathbb{C}, \mathbb{R}, \mathbb{G}_m$[a].

[a] In this lecture, we will never use $\mathbb{Q}$ actually.

In the end, we consider the complete reducibility of $\mathbb{T}_n$. Recall that there is an $\mathbb{T}_n$-action on $\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$: for $f \in M$, we define $f_g(x) = f(g \cdot x)$ for $x \in \mathbb{T}_n$. It is clear that $\mathbb{C}\chi_a = \{c\chi_a : c \in \mathbb{C}\}$ is an invariant subspace of $\mathbb{C}[M]$ for all χ_a, and it is the eigenspace in the sense $\mathbb{C}\chi = \{f \in \mathbb{C}[M] : f_g = \chi_a(g) f, \forall g \in \mathbb{T}_n\}$.

We have the following property

Proposition 1.4 ([CLS11, Proposition 1.1.16])**.** (Exercise 1.1) *For an $\mathbb{T}_n$ invariant subspace $A \subset \mathbb{C}[M]$, we have the following eigenspace decomposition*

$$A = \bigoplus_{\chi_a \in A} \mathbb{C}\chi_a.$$

Remark and Hints: It is cleat that the right hand side is a subspace of the left hand side. The point of the proof is that the inclusion is an equal, and essentially, the non-trivial point here is that it is a direct sum rather than direct product. This comes from the fact that the action $\mathbb{T}_n \times \mathbb{T}_n \to \mathbb{T}_n$ is algebraic, which means that there exists a ring homomorphism $\mathbb{C}[M] \to \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M]$ and the right hand side is a "polynomial ring".

2. Toric variety

Definition 2.1. A toric variety is an algebraic variety X satisfies the following conditions:
(1) X is irreducible, separated and normal.[1]
(2) There exists an effective algebraic action $\mathbb{T}_n \times X \to X$. (Effective means that $\mathbb{T}_n \to \mathrm{Aut}(X)$ is injective.)
(3) There exists $x \in X$ such that its orbit $\mathbb{T}_n \cdot x$ is isomorphic to $\mathbb{T}_n$ and form an open dense subset of X w.r.t Zariski topology.

As we mentioned in the footnote 1, we will mainly focus on the condition (2) and (3) here.

Example 2.2. (1) For $X = \mathbb{C}^n$, we have the standard diagonal tori action on $\mathbb{C}^n$, which is effective. An open dense tori orbit can be given by $\mathbb{T}_n = \mathbb{T}_n \cdot (1, \cdots, 1)$.
(2) For $X = \mathbb{P}^n$, we have the tori action on $\mathbb{P}^n$ via $(z_1, \cdots, z_n) \cdot [x_0, x_1, \cdots, x_n] = [x_0, z_1 x_1, \cdots, z_n x_n]$, which is effective. An open dense tori orbit can be given by $\mathbb{T}_n = \mathbb{T}_n \cdot [0, 1, \cdots, 1]$.
(3) Product of toric varieties is toric.

Toric variety has an open dense tori, so they are all birational equivalent. In particular, we have

Proposition 2.3. *The field of rational functions (in the fancy sense) on a toric variety is the field of rational functions (in the very traditional sense).*

Proof. As we have seen that any toric variety are birational equivalent. In particular, birational equivalent to $\mathbb{C}^n$. Then rational function field of toric variety is isomorphic to rational function field of $\mathbb{C}^n$, which is the field of fraction of of polynomials. $\square$

The main structure theorem is the following

Theorem 2.4 (Sumihiro). *For a toric variety X, there exists a (finite[2]) $\mathbb{T}_n$-equivariant open cover where open sets are affine toric varieties.*

It motives us to classify affine toric varieties.

Example 2.5. We know that $\mathbb{P}^n = \cup_{i=0}^n \mathbb{C}_i^n$ where $\mathbb{C}_i^n = \mathbb{P}^n \setminus \{x_i = 0\}$ is an affine toric cover.

[1]Under classical topology, irreducible basically means connected, separated basically means Hausdorff. Normal is more tricky, which basically means that for any point, there is an connected punctured neighborhood; in fact, under the classical topology, normal varieties are stratified pseudomanifolds. Normality implies that singularity occurs only in codimension 2, and certain Hartogs extension exists. In this course, we only consider normal toric varieties, and you can skip all discussion about normality if you don't know what it actually means.

[2]This finiteness is a general property for algebraic varieties.

3. Affine toric variety

In this section, we classify all affine toric varieties. For X toric, we assume $X = \mathrm{Spec}(A)$ for an commutative algebra A. Then A is the ring of polynomial functions on X.

By definition of toric variety, $\mathbb{T}$ is an Zariski open dense set of X, we have that for any polynomial function $f \in A$ (that is continuous w.r.t. Zariski topology), it is determined by its restriction on $\mathbb{T}$. Therefore, we have that A can be view as an subalgebra of $\mathbb{C}[M]$ where M is the character lattice of $\mathbb{T}$ via the open inclusion $\mathbb{T} \subset X$.

Moreover, $\mathbb{T}$ is an $\mathbb{T}$-invariant open set of X, we have that A is $\mathbb{T}$-invariant subalgebra of $\mathbb{C}[M]$. Then by Proposition 1.4, we have that there exists a set

$$S = \{\chi_m \in M : \chi_m \in A\}$$

such that $A = \oplus_{\chi_m \in A} \mathbb{C}\chi_m$.

Example 3.1. (1) For $X = \mathbb{T}$, it is clear by definition of Laurent polynomial ring that $S = M$.
(2) For $X = \mathbb{C}^n$, then we know that $A = \mathbb{C}[x_1, \cdots, x_n]$. It is clear by definition of polynomial ring that $S \simeq \mathbb{N}^n$.

Here, one feature is that S should be a sub-monoid of M. In fact, we have

Proposition 3.2. (Exercise 3.1) *The set S define above satisfying the following properties:*
(1) $1 \in S$ and if $\chi_m, \chi_n \in S$, then $\chi_{m+n} = \chi_m\chi_n \in S$.
(2) If $\chi_m^k \in S$, then $\chi_m \in S$.
(3) S is finitely generated as a monoid.

In the proposition, we use the multiplicative notation for the abelian group structure. Under an identification $M \simeq \mathbb{Z}^n$, we can also think S as an additive sub-monoid of $\mathbb{Z}^n$. In the future, we will basically only use this additive convention.

By Proposition 3.2-(3), we can assume that

$$S = \mathbb{N}\chi_1 + \cdots \mathbb{N}\chi_r \subset M.$$

We set[3]

$$\tau = \mathrm{Cone}(\chi_1, \cdots, \chi_r) = \{\sum a_i\chi_i : a_i \geq 0\} \subset M_{\mathbb{R}}.$$

(⚠ This paragraph contains some definitions.) Then we know that τ is a *rational*, *polyhedral* convex cone, and $S = \tau \cap M$ is saturated in the sense of Proposition 3.2-(2). Here, rational means that it is generated by integer vectors with respect to a fixed lattice M (so it is better to say it is M-rational); and polyhedral means that τ is given by finite intersection of half-spaces.

In summary, we have

Theorem 3.3. *If X is an affine toric variety, then there exists an rational polyhedral convex cone $\tau \subset M_{\mathbb{R}}$ such that*

$$X = \mathrm{Spec}(\mathbb{C}[\tau \cap M]).$$

Here, $\mathbb{C}[\tau \cap M]$ means the monoid ring generated by the monoid $\tau \cap M$.

Conversely, we have

Theorem 3.4. (Exercise 3.2) *For any rational polyhedral cone $\tau \subset M_{\mathbb{R}}$ such that $\tau \cap M$ is saturated (saturation implies normality). Then*

$$X^\tau = \mathrm{Spec}(\mathbb{C}[\tau \cap M]).$$

is an affine toric variety.

Remark 3.5. The theorem basically says that, if $\chi \in \tau \cap M$, then χ can be extended to a polynomial function $\widetilde{\chi}$ on X^τ whose restriction on $\mathbb{T}$ is χ.

[3] Convention: $\mathrm{Cone}(\varnothing) := \{0\}$.

These two theorems completely classify affine toric varieties.

We also notice the following relation: if $\tau_1 \subset \tau_2$ is a inclusion of rational polyhedral cones (we require saturation), then there exists an algebraic morphism $X^{\tau_2} \to X^{\tau_1}$.

Using the description of M, we learn information about polynomial functions on affine toric varieties. We also need a dual description on N to describe 1-parameter subgroups in affine toric varieties. Recall that there exists a perfect pairing between M and N. Then for a cone $\sigma \subset N_{\mathbb{R}}$, we can define

$$\sigma^\vee := \{m \in M_{\mathbb{R}} : \langle m, n\rangle \geq 0, \forall n \in N_{\mathbb{R}}\} \subset M_{\mathbb{R}}.$$

Example 3.6. Here, we draw σ in blue and draw $\sigma^\vee$ in green.

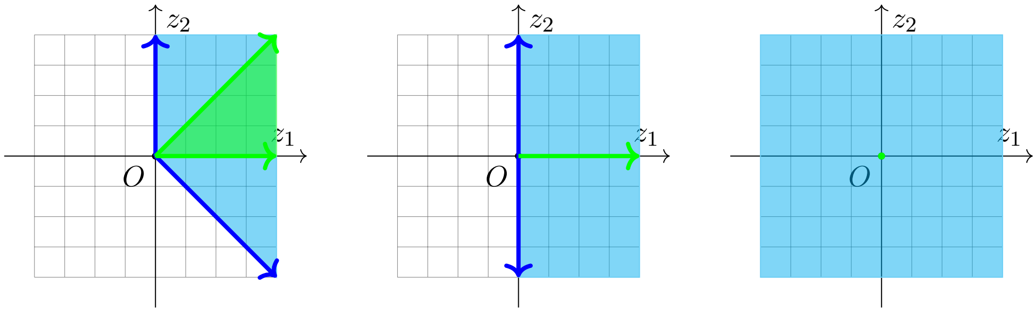


FIGURE 1. Cones and dual cones

We have the following simple linear algebra lemma

Lemma 3.7. *For a N-rational, polyhedral convex cone $\sigma \subset N_{\mathbb{R}}$, we have:*

(1) $\tau = \sigma^\vee$ is M-rational and polyhedral;
(2) $(\sigma^\vee)^\vee = \tau^\vee = \sigma$;
(3) For $\tau = \sigma^\vee$, $\tau \cap M$ is saturated if and only if σ is strongly convex (i.e. $\sigma + (-\sigma) = \{0\}$, or equivalently, there is no line in σ).

We the notation above, we introduce the notation

$$X_\sigma := X^{\sigma^\vee} = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M]).$$

Then we have the following property

Proposition 3.8. *For a N-rational, polyhedral convex cone $\sigma \subset N_{\mathbb{R}}$ and $\lambda \in N$, we define $\widetilde{\lambda} : \mathbb{C}^\times \xrightarrow{\lambda} \mathbb{T} \to X_\sigma$. Then we have that $\widetilde{\lambda}$ extends to a curve $\widetilde{\lambda} : \mathbb{C} \to X_\sigma$ if and only if $\lambda \in \sigma$.*

Proof. For $\chi \in M$, we have

$$\chi(\widetilde{\lambda}(t)) = t^{\langle \chi, \lambda\rangle}.$$

So, $\lim_{t\to 0} \widetilde{\lambda}(t)$ exists in X_σ if and only if $\langle \chi, \lambda\rangle \geq 0$ for all $\chi \in \sigma^\vee$, which is equivalent to that $\lambda \in (\sigma^\vee)^\vee = \sigma$. □

Lastly, we consider smoothness of affine toric varieties. Strongly convex N-rational, polyhedral convex cone $\sigma \subset N_{\mathbb{R}}$ can be generated by a canonical set of generators: Any edge ρ of σ is a ray (i.e. linearly equivalent to $\mathbb{R}_{\geq 0}$) since there is no lines in σ. Then $\rho \cap N \simeq \mathbb{N}u_\rho$ for a unique u_ρ. We call u_ρ the ray generator of ρ. We have the following combinatorial lemma

Lemma 3.9. *Strongly convex N-rational, polyhedral convex cone $\sigma \subset N_{\mathbb{R}}$ is generated by its ray generators of its edges.*

We call the ray generators of edges the minimal generators of σ.

Now, we state the theorem without proof (even not for exercise).

Theorem 3.10. *For a strongly convex rational polyhedral convex cone $\sigma \subset N_{\mathbb{R}}$, we have*

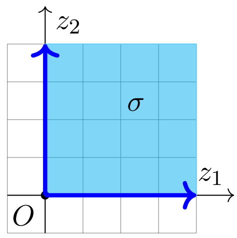
(1) X_σ is smooth if and only if $\sigma \cap N$ is smooth in the sense: the minimal generators of σ form a $\mathbb{Z}$-basis of N.

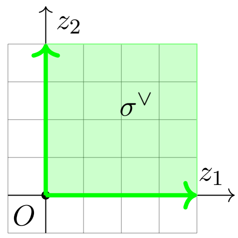
(2) X_σ has orbifold singularities if and only if σ is simplicial in the sense: the minimal generators of $\sigma \cap N$ form a $\mathbb{Z}$-linearly independent set of N.

Summary: You can randomly pick a strongly convex rational polyhedral cone $\sigma \subset N_\mathbb{R}$, then you can obtain a unique affine toric variety X_σ: by Lemma 3.7, $\sigma^\vee \cap M$ is saturated, then you can apply Theorem 3.4 to obtain X_σ, which is unique by Theorem 3.3. Then you can test X_σ is smooth or not by study the minimal generator of σ.

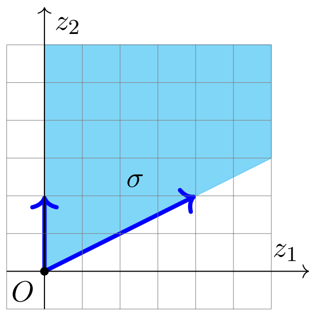
In the rest of this section, we make some example to test.

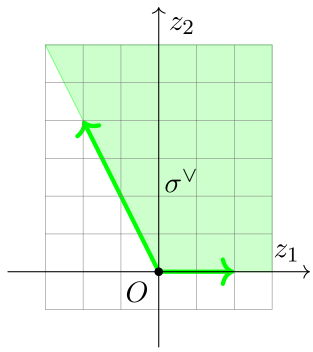
Example 3.11. (1) Consider $\sigma = \mathbb{R}^2_{\geq 0}$ in $N = \mathbb{Z}^2$, we have $\sigma^\vee = \mathbb{R}^2_{\geq 0}$ as well. Then $\sigma^\vee \cap M = \mathbb{N}^2$ and $X_\sigma = \mathrm{Spec}(\mathbb{C}[\mathbb{N}^2]) = \mathbb{C}^2$.





(2) Consider the following cones. Where thick arrow means minimal generators So, we can see



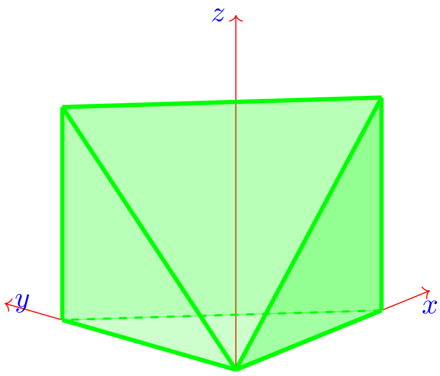


that
$$\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[x_1, x_2^2/x_1] \simeq \frac{\mathbb{C}[u,v,w]}{\langle uw = v^2 \rangle}$$
by change of coordinates $u = x_1, v = x_2$ and $w = x_2^2/x_1$.

Therefore, X_σ is the cone singularity (or A_1-singularity) $\mathbb{C}^2/\mathbb{Z}_2$, which is a orbifold singularities. This is clear from our combinatorial criterion: the minimal generators $(2,1),(0,1)$ of σ are $\mathbb{Z}$-linear independent but not a base (they cannot generate $(1,0)$ for example).

(3) For $N = \mathbb{Z}^3$, pick $e_1, e_2, e_1+e_3, e_2+e_3$ as minimal generators to generate a cone. One can check that
$$\sigma^\vee = \mathrm{Cone}(e_1, e_2, e_3, e_1+e_2-e_3).$$
Then we have
$$\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[x_1, x_2, x_3, x_1x_2/x_3] \simeq \frac{\mathbb{C}[u,v,w,z]}{\langle uv = wz \rangle}$$
by change of coordinates $u = x_1, v = x_2$, $w = x_3$ and $z = x_1x_2/x_3$. $\{uv = wz\}$ is the conifold singularity, which, as kind of common sense, known not to be a orbifold singularity. It can be tested by our combinatorial criterion since $e_1, e_2, e_1+e_3, e_2+e_3$ do not even form a $\mathbb{Z}$-linearly independent set.

(4) (Exercise 3.3) For two strongly convex polyhedral cones $\sigma_1 \subset (N_1)_\mathbb{R}$, $\sigma_2 \subset (N_2)_\mathbb{R}$, we have
$$X_{\sigma_1 \times \sigma_2} \simeq X_{\sigma_1} \times X_{\sigma_2}.$$



In the rest of the section, we study toric morphism between affine toric varieties.

(Exercise 3.4) For two strongly convex polyhedral cones $\sigma_1 \subset (N_1)_{\mathbb{R}}$, $\sigma_2 \subset (N_2)_{\mathbb{R}}$, and a $\mathbb{Z}$-module morphism $\overline{\phi} : N_1 \to N_2$. Then we we have an algebraic group homomorphism $\phi = \overline{\phi} \otimes \mathbb{C}^{\times} : \mathbb{T}_1 \to \mathbb{T}_2$ (notice here, subscript does not mean dimension, only indicates that there are two things). Prove that ϕ extends to a ϕ-equivariant morphism $\phi : X_{\sigma_1} \to X_{\sigma_2}$ if and only if $\overline{\phi} \otimes \mathbb{R}(\sigma_1) \subset \sigma_2$. We shall call equivariant morphisms defined in this way toric morphisms.

Let us give an example for it.

Example 3.12. Here, we notice that we did not fix identifications $N = \mathbb{Z}^n$ in general. We produce an example using this feature.

We take $N_1 = (2\mathbb{Z})^2$ and $N_2 = \mathbb{Z}^2$. However, we have $(N_1)_{\mathbb{R}} = (N_2)_{\mathbb{R}} = \mathbb{R}^2$. Now, we take $\sigma_1 = \sigma_2 = \mathbb{R}^2_{\geq 0}$ as Example 3.11-(1). It is clear that both of affine toric varieties are $\mathbb{C}^2$.

Then Exercise 3.4 shows that the inclusion map $N_1 \subset N_2$ induces a toric morphism $\phi : \mathbb{C}^2 \to \mathbb{C}^2$. We check that the map is not identity.

In fact, we should have $M_1 = (\mathbb{Z}/2)^2$ and $M_2 = \mathbb{Z}^2$. Then $\sigma_1^{\vee} \cap M_1 = (\mathbb{N}/2)^2$ and $\sigma_2^{\vee} \cap M_2 = \mathbb{N}^2$. Here, we identify x_i with $1/2e_i$ in forming monoidal algebra, then we have

$$\mathbb{C}[\sigma_1^{\vee} \cap M_1] = \mathbb{C}[x_1, \cdots, x_n], \quad \mathbb{C}[\sigma_2^{\vee} \cap M_2] = \mathbb{C}[x_1^2, \cdots, x_n^2].$$

Then we know that the inclusion map $N_1 \subset N_2$ induces

$$\phi : \mathbb{C}^2 \to \mathbb{C}^2, \quad (x_1, \cdots, x_n) \mapsto (x_1^2, \cdots, x_n^2).$$

4. Fan and toric variety

Recall the Sumihiro Theorem 2.4. It tells that we can always find an affine toric open cover. It encourage us to glue affine toric varieties with certain combinatorial data given.

Definition 4.1. A fan (N, Σ) is pair consist of a lattice N and a finite set Σ of subsets of $N_{\mathbb{R}}$ such that

(1) If $\sigma \in \Sigma$, then σ is a N-rational strongly convex polyhedral cones in $N_{\mathbb{R}}$.
(2) For $\sigma \in \Sigma$, and if τ is a face of σ (where we denoted by $\tau \prec \sigma$), then $\tau \in \Sigma$ [4].
(3) For $\sigma_1, \sigma_2 \in \Sigma$, we have $\sigma_1 \cap \sigma_2$ is a face for both σ_i. (Then $\sigma_1 \cap \sigma_2$ is in Σ by (2)).

When N is clear, we often say Σ a fan.

So, the full classification theorem for toric variety is

Theorem 4.2. *Toric varieties are 1-1 correspondences to fans.*

We refer to [CLS11, Section 3.1] for more details. In this section, I will explain how to construct a toric variety from a fan (this part is is useful). The other direction need more technical tool that we will explain in Section 5 (but it is clear that if you want to use toric varieties in your research, this direction is useless.)

Remark 4.3. Let us pay attention to affine toric variety. For an strongly convex N-rational polyhedral cone σ, we can associate X_σ, then what is the fan of it as predicted here? The answer is (N, Σ) that produce X_σ is exactly given by the fan spanned by all faces of σ. You can verify this by hand.

Example 4.4. Here, we consider two examples of fans with $N = \mathbb{Z}^2$.

(1) We have $\Sigma = \{\{0\}, \mathbb{R}_{\geq 0}e_1, \mathbb{R}_{\geq 0}e_2, \mathbb{R}_{\geq 0}(e_1 + e_2), \sigma_1, \sigma_2\}$.
(2) We have $\Sigma = \{\{0\}, \mathbb{R}_{\geq 0}e_1, \mathbb{R}_{\geq 0}e_2, \mathbb{R}_{\geq 0}(-e_1 - e_2), \sigma_1, \sigma_2, \sigma_3\}$.

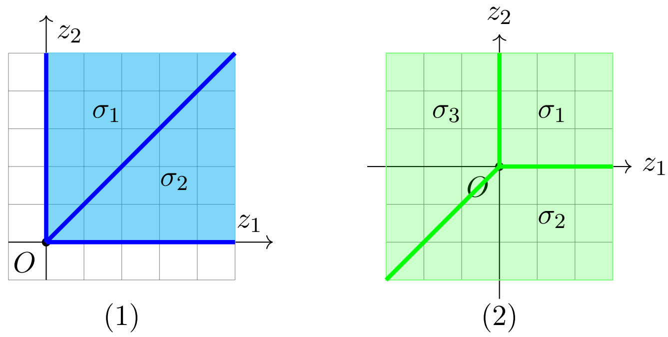


Figure: In these two figures, we only mark top dimension cones. While, rays and the origin are also cones in these fans.

(3) This example will explain the importance of specifying N. We set $u_1 = (2, 1), u_2 = (0, 1)$. We consider the cone $\sigma = \mathrm{Cone}(u_1, u_2) \subset \mathbb{R}^2$. We set $\Sigma = \{\{0\}, \mathbb{R}_{\geq 0}u_1, \mathbb{R}_{\geq 0}u_2, \sigma\}$.

Now, we take $N_1 = \mathbb{Z}^2$ and $N_2 = \mathbb{Z}u_1 \oplus \mathbb{Z}u_2$. Then N_2 is a sublattice of N_1, and $(N_2)_{\mathbb{R}} = (N_1)_{\mathbb{R}} = \mathbb{R}^2$. So, we can think (N_1, Σ) and (N_2, Σ) as two fans. Since these two fans are associated with cone, we can see that both of them are affine. However, by Example 3.11-(1) and (2), we see that $X_{(N_2,\Sigma)} \simeq \mathbb{C}^2$ is smooth and $X_{(N_1,\Sigma)} \simeq \mathbb{C}^2/(\mathbb{Z}_2)$ is the A_1-singularity.

Now, we define a toric variety using a fan. First of all, for each $\sigma \in \Sigma$, we have construed an affine toric variety $X_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$. To emphasize it is an open set in a general toric variety, we use U_σ to denote X_σ here.

To glue them, we should understand relation for two fans with a common face.

[4] It makes sense since σ is a cone polyhedral

Lemma 4.5 (Exercise 4.1)**.** *For two cones in $\sigma_1, \sigma_2 \in \Sigma$, we set $\tau = \sigma_1 \cap \sigma_2$ ($\in \Sigma$ by definition of fans). Then we have*

(1) There exists an $m \in \sigma_1^\vee \cap (-\sigma_2^\vee) \cap M = (\sigma_1 - \sigma_2)^\vee \cap M$ such that $\sigma_1 \cap H_m = \tau = \sigma_2 \cap H_{-m}$ where $H_m = H_{-m} = \{n \in N_{\mathbb{R}} : \langle m, n\rangle = 0\}$ is a hyperplane.
(2) With the notation above, we have $\tau^\vee \cap M = \sigma_1^\vee \cap M + \mathbb{Z}(-m)$.
(3) With the notation above, we have $\tau^\vee \cap M = \sigma_1^\vee \cap M + \sigma_2^\vee \cap M$.

Therefore, by Lemma 4.5-(2), we have

$$U_\tau = \mathrm{Spec}(\mathbb{C}[\sigma_1^\vee \cap M + \mathbb{Z}(-m)]) \simeq \mathrm{Spec}(\mathbb{C}[\sigma_1^\vee][\frac{1}{\chi_m}]) \simeq U_{\sigma_1} \cap \{\chi_m \neq 0\} \overset{open}{\subset} U_{\sigma_1}.$$

So, we define the isomorphism of varieties by Lemma 4.5-(3)[5]

$$g_{21} : U_{\sigma_1} \cap \{\chi_m \neq 0\} \simeq U_\tau \simeq U_{\sigma_2} \cap \{\chi_{-m} \neq 0\}.$$

There is a tautological but tedious verification to show that, for $\sigma_i \in \Sigma$ where $i = 1, 2, 3$, we have the cocycle condition, when the following maps can be defined:

$$g_{21} = g_{12}^{-1}, \quad g_{31} = g_{32} g_{21}.$$

Then we define an equivalent relation on $\bigsqcup_\sigma U_\sigma$ such that $(\sigma_1, x_1) \simeq (\sigma_2, x_2)$ when $x_2 = g_{21}(x_1)$. The cocycle conditions gurantee that this is indeed an equivalent relation.

Definition 4.6. We define

$$X_\Sigma := \bigsqcup_\sigma U_\sigma / \simeq .$$

Theorem 4.7. *We have that X_Σ is a toric variety.*

Proof. It is clear that, for each σ, U_σ is an Zariski open set, which is affine and toric. We skip verification for irreducibility and separating. Normality is local, so it follows from normality of U_σ.

Now, we can check that g_{ij} are $\mathbb{T}$-equivariant, then we can glue all $\mathbb{T}$-action on each U_σ. Each U_σ has an open dense tori, then they are also open dense torus of X_Σ since Zariski open sets in an irreducible variety are always dense. $\square$

Example 4.8. (1) Consider the fan given in Example 4.4-(2): $\Sigma = \{\{0\}, \tau_2 = \mathbb{R}_{\geq 0} e_1, \tau_1 = \mathbb{R}_{\geq 0} e_2, \tau_3 = \mathbb{R}_{\geq 0}(-e_1 - e_2), \sigma_1, \sigma_2, \sigma_3\}$. Now we explain that it gives $X_\Sigma = \mathbb{P}^2$.

We have the following

$$\begin{aligned}
U_{\sigma_1} &= \mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[x, y]) \\
U_{\sigma_2} &= \mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[xy^{-1}, y^{-1}]) \\
U_{\sigma_3} &= \mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[x^{-1}, x^{-1}y]) \\
U_{\tau_1} &= \mathbb{C}^\times \times \mathbb{C} = \mathrm{Spec}(\mathbb{C}[x^{\pm 1}, y]) \\
U_{\tau_2} &= \mathbb{C} \times \mathbb{C}^\times = \mathrm{Spec}(\mathbb{C}[x, y^{\pm 1}]) \\
U_{\tau_3} &= \mathbb{C} \times \mathbb{C}^\times = \mathrm{Spec}(\mathbb{C}[(xy)^{-1}, (x^{-1}y)^{\pm 1}]) \\
U_{\{0\}} &= \mathbb{C}^\times \times \mathbb{C}^\times = \mathrm{Spec}(\mathbb{C}[x^{\pm 1}, y^{\pm 1}])
\end{aligned}$$

Let's do computation for g, consider σ_1, σ_3 for example: They intersect at τ_1. We naturally have identifications of rings

$$\mathbb{C}[x, y][\frac{1}{x}] = \mathbb{C}[x^{\pm 1}, y] = \mathbb{C}[x^{-1}, x^{-1}y][\frac{1}{x^{-1}}]$$

which induces

$$U_{\sigma_1} \cap \{x \neq 0\} = U_{\tau_1} = U_{\sigma_2} \cap \{x^{-1} \neq 0\}.$$

[5]In fact, with good choice of coordinates, you may find that many of g here are identity maps. Heuristically, this is because all identifications here are quite natural

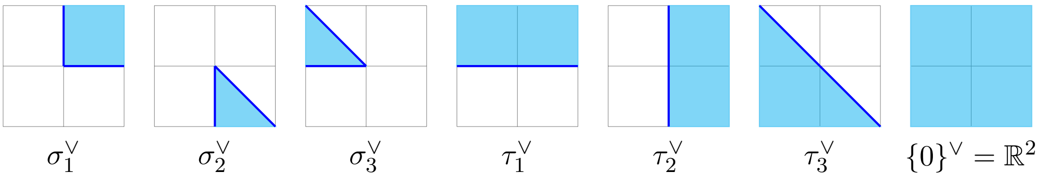


Figure: Dual cones of the $\mathbb{P}^2$ fan.

Here, you may confuse of what are x, y here. In fact, consider the homogeneous coordinate $[x_0, x_1, x_2] \in \mathbb{P}^2$, we can actually find that $x = x_1/x_0, y = x_2/x_0$. It is easy to match that

$$U_{\sigma_1} = \mathbb{P}^2 \setminus \{x_0 \neq 0\}, \quad U_{\sigma_2} = \mathbb{P}^2 \setminus \{x_2 \neq 0\}, \quad U_{\sigma_1} = \mathbb{P}^2 \setminus \{x_1 \neq 0\}.$$

And the transition functors g is the same as those we defining $\mathbb{P}^2$. Then we can naturally identify $\mathbb{P}^2 = X_\Sigma$.

In general, consider the following fan Σ in $\mathbb{Z}^n$ defines $\mathbb{P}^n$: We set $e_0 = -e_1 - \cdots - e_n$, and

$$\Sigma = \{\text{Proper subsets of } \{e_0, e_1, \cdots, e_n\}\}.$$

(2) In fact, we can classify all fans of dimension 1: Except fans come from cone: $\{0\} \leftrightarrow \mathbb{C}^\times$ and $[0, \infty) \leftrightarrow \mathbb{C}$. The only new one is given by the $\mathbb{P}^1$ fan: $\{[0, \infty), (-\infty, 0], \{0\}\}$.

(3) (Exercise 4.2): Consider the following fan. Prove that the corresponding toric variety is the Hirzebruch surface H_2. (You can try either for just $r = 2$ or for general r.) In general, if we change the ray generator $(-1, 2)$ to $(-1, r)$, we get the Hirzebruch surface H_r, i.e. the total space of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-r)) \to \mathbb{P}^1$ over $\mathbb{P}^1$.

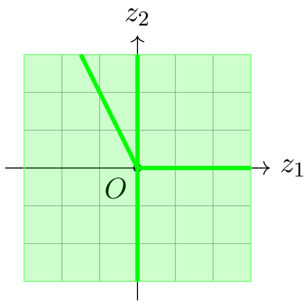


Figure: Fan of the Hirzebruch surface H_2

(4) Let $(q_0, q_1, \cdots, q_n) \in \mathbb{N}^{n+1}$ with $\gcd(q_0, q_1, \cdots, q_n) = 1$. Consider $N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0, q_1, \cdots, q_n)$. We set u_i the natural projection of $e_i \in \mathbb{Z}^{n+1}$ onto N, and

$$\Sigma = \{\text{Proper subsets of } \{u_0, u_1, \cdots, u_n\}\}.$$

Then we have $X_\Sigma = \mathbb{P}^n(q_0, q_1, \cdots, q_n)$, the weighted projective space.

(5) (Exercise 4.3): For two fans Σ_1 and Σ_2, we define

$$\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 : \sigma_i \in \Sigma_i,\ i = 1, 2\}.$$

Prove that i) $\Sigma_1 \times \Sigma_2$ is a fan. ii) $X_{\Sigma_1 \times \Sigma_2} \simeq X_{\Sigma_1} \times X_{\Sigma_2}$.

5. Orbit-Cone correspondence

Now, we discuss the Orbit-Cone correspondence (OCC for short) for toric varieties. It is a crucial technical result.

To begin with, we start from defining a specific (closed) point on an affine toric variety.

For a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, we have the following 1-1 correspondences

$$\begin{gathered}\{\text{(closed) points in } X_\sigma\}\\ \updownarrow \\ \{\mathbb{C}\text{-algebra homomorphisms: } \mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}\} \\ \updownarrow \\ \{\text{Monoid homomorphisms: } \sigma^\vee \cap M \to \mathbb{C}\}.\end{gathered}$$

The first one is the Yoneda, and the second one comes from the universal property of monoidal algebra.

Therefore, we define a point p_σ by the monoidal homomorphism $\sigma^\vee \cap M \to \mathbb{C}$:

$$m \mapsto 1, m \in \sigma^\perp \cap M; \quad m \mapsto 0, m \in \sigma^\vee \cap M \setminus \sigma^\perp.$$

Geometrically, we can characterize the point p_σ in the following way

Proposition 5.1 (A precise version of Proposition 3.8)**.** *For a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, and $b \in \mathrm{RelInt}(\sigma) \cap N$ (footnote here [6]). Then we have $\lim_{t\to 0} \widetilde{\lambda_b}(t) = p_\sigma$.*

Proof. By Proposition 3.8, the limit always exists since $\mathrm{RelInt}(\sigma) \subset \sigma$. As we computedin Proposition 3.8, for every $m \in \sigma^\vee \cap M$, we have $\langle m, \lim_{t\to 0} \widetilde{\lambda_b}(t)\rangle = \lim_{t\to 0} t^{\langle m,b\rangle}$. Therefore, if $b \in \mathrm{RelInt}(\sigma)$, which actually means that $\langle m, b\rangle > 0$ for $m \in \sigma^\vee \cap M \setminus \sigma^\perp$, we have that $\lim_{t\to 0} t^{\langle m,b\rangle} \to 0$. Similarly, we prove that for $m \in \sigma^\perp \cap M$, we have $\langle m, b\rangle = 0$ and $\lim_{t\to 0} t^{\langle m,b\rangle} \to 1$.

This exactly means that $\lim_{t\to 0} \widetilde{\lambda_b}(t) = p_\sigma$. □

Remark 5.2. In fact, this proposition is enough to determine all possible limits: If b is not in the relative interior of σ, then it must means that it is in a face τ of σ. So, we can ask again that if b is in the relative interior of τ, and then continuous. This process will stop in finite steps since our cones are all finite dimensional.

Now, we state the following theorem without proof. We refer to [CLS11, Section 3.2] for its proof. (Actually, it is not hard, just long.)

Theorem 5.3 (Orbit-Cone correspondence)**.** *For a fan Σ and its toric variety X_Σ, we have:*

(1) There is a 1-1 correspondence

$$\{\textit{Cones in } \Sigma\} \leftrightarrow \{\mathbb{T}-\textit{orbits in } X_\Sigma\}, \quad \sigma \mapsto O(\sigma) := \mathbb{T}\cdot p_\sigma.$$

(2) We have

$$\dim O(\sigma) + \dim \sigma = \dim N_{\mathbb{R}}.$$

The first dimension is the Krull dimension of varieties, the second is the dimension of the vector space spaned by σ, the third is the dimension of vector spaces.

(3) We have

$$U_\sigma = \bigcup_{\tau \prec \sigma} O(\tau).$$

(4) We have

$$\tau \prec \sigma \iff O(\sigma) \subset \overline{O(\tau)}$$

and

$$\overline{O(\tau)} = \bigcup_{\tau \prec \sigma} O(\sigma).$$

Here the closure in both Zariski topology and analytic topology are the same.

[6] Here, if $\sigma \simeq \mathbb{R}^m_{\geq 0} \subset \mathbb{R}^n$ $(0 < m \leq n)$, we mean that $\mathrm{RelInt}(\sigma)$ is given by $\mathbb{R}^m_{>0}$; we also set $\mathrm{RelInt}(\{0\}) = \{0\}$

Example 5.4. Again, let us test the result using $\mathbb{P}^2$: recall the fan given in Example 4.4-(2): $\Sigma = \{\{0\}, \tau_2 = \mathbb{R}_{\geq 0}e_1, \tau_1 = \mathbb{R}_{\geq 0}e_2, \tau_3 = \mathbb{R}_{\geq 0}(-e_1 - e_2), \sigma_1, \sigma_2, \sigma_3\}$.

We have the following table: for b in the relative interior oif the cone of in first row, the second row gives the corresponding limit $\lim_{t\to 0} \lambda^u(t)$. Then it is easy to describe all possible

Cones	σ_1	σ_2	σ_3	τ_1	τ_2	τ_3	$\{0\}$
$\lim_{t\to 0} \lambda^b(t)$	$[1,0,0]$	$[0,0,1]$	$[0,1,0]$	$[1,1,0]$	$[1,0,1]$	$[0,1,1]$	$[1,1,1]$

$\mathbb{T}_2$-orbits. Let me do some specific examples to illustrate:

- For $b = (2,1) \in \mathrm{RelInt}(\sigma_1)$, $\lambda^b(t) = [1, t^2, t] \to [1,0,0]$. Then $O(\sigma_1) = [1,0,0]$ is a fixed point (which can be seen by OCC since σ_1 is 2 dimension).
- For $b = (2,0) \in \mathrm{RelInt}(\tau_2)$, $\lambda^b(t) = [1, t^2, 1] \to [1,0,1]$. Then $O(\tau_2) = \{[1, z, 0] : z \neq 0\} \simeq \mathbb{C}^\times$.
- For $b = (0,0) \in \mathrm{RelInt}(\{0\})$, $\lambda^b(t) = [1,1,1] \to [1,1,1]$. Then $O(\{0\}) = \{[1, z_1, z_2] : z_i \neq 0\} \simeq (\mathbb{C}^\times)^2$ is the big tori.

Here, we state the following theorem.

Theorem 5.5. *For a fan Σ, we have*

(1) X_Σ is smooth (orbifold) if and only if $\forall \sigma \in \Sigma$, σ is smooth (simplicial).
(2) X_σ is compact if and only if the support of the fan $|\Sigma| = \cup_{\sigma\in\Sigma}\sigma$ equals $N_\mathbb{R}$.

Proof. For the first one, the conclusion is local. So follows from the affine result Theorem 3.10.

For the second statement, from compactness to the supportive follows from Proposition 5.1. The other direction is harder and need OCC Theorem 5.3, we skip it. Try to convince yourself from examples (above and below). □

(Exercise 5.1): Use Proposition 5.1 to show one direction of Theorem 5.5-(2): if X_Σ is compact (or complete in algebraic geometry), then $|\Sigma| = \cup_{\sigma\in\Sigma}\sigma = N_\mathbb{R}$.

We finish this section by considering the following result on a precise form of singularity distribution, which is left as (Exercise 5.2)

Proposition 5.6 ([CLS11, Proposition 11.1.2])**.** *For a toric variety of fan Σ, we have*

$$(X_\Sigma)_{sing} = \bigcup_{\sigma \text{ is not smooth}} \overline{O(\sigma)}, \quad (X_\Sigma)_{reg} = \bigcup_{\sigma \text{ is smooth}} U_\sigma.$$

6. Toric morphism

In fact, we have discuss toric morphism in affine situation in Exercise 3.4. As usual, the general case is just pinch together all local information.

For two fans (together with lattices): (Σ_1, N_1) and (Σ_2, N_2) and a $\mathbb{Z}$-linear map $\overline{\phi} : N_1 \to N_2$. We say $\overline{\phi}$ is compatible with two fans if $\forall \sigma_1 \in \Sigma_1$, $\exists \sigma_2 \in \Sigma_2$ such that $\overline{\phi} \otimes \mathbb{R}(\sigma_1) \subset \sigma_2$.

Proposition 6.1. *If $\overline{\phi}$ is compatible with (Σ_1, N_1) and (Σ_2, N_2), then $\overline{\phi} \otimes \mathbb{C}^\times$ extends to an ($\overline{\phi} \otimes \mathbb{C}^\times$-)equivariant algebraic map $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$.*

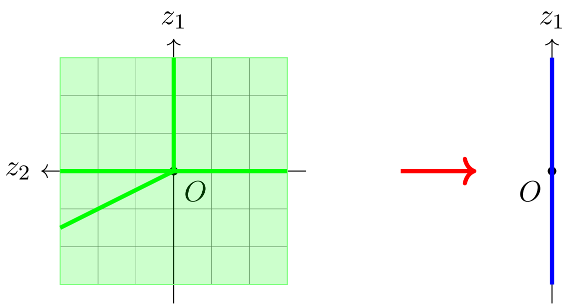
Proof. This is just the global version of one direction of (Exercise 3.4). □

Proposition 6.2. *If an equivariant map $f : X_{\Sigma_1} \to X_{\Sigma_2}$ restricts to a group homomorphism $\mathbb{T}_1 \to \mathbb{T}_2$. Then there exists a $\overline{\phi}$ that is compatible with fans such that $f = \phi$.*

It is also the other direct of the global version of (Exercise 3.4). But here, you need use OCC to clarify some technical discussion.

For detail proofs of above two propositions, we refer to [CLS11, Section 3.3]. We call those morphisms ϕ a toric morphism.

Example 6.3. (1) For $\ell \in \mathbb{N}$ and $\overline{\phi}_\ell : N \to N$, $\overline{\phi}_\ell(n) = \ell n$. It induces a ramified ℓ-cover of X_Σ to itself, which is possible ramified along the toric boundary $X_\Sigma \setminus \mathbb{T}$.
(2) Consider the following compatible map from the Hirzebruch fan to $\mathbb{P}^1$ fan (the linear map $\overline{\phi}$ is just a projection to the first factor). We claim that this is actually the bundle projection of Hirzebruch surface $H_r \to \mathbb{P}^1$. (Recall that H_r is defined as the total space of the $\mathbb{P}^1$ bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-r)) \to \mathbb{P}^1$ over $\mathbb{P}^1$.



(3) (Exercise 6.1) Sublattice and finite group quotient ([CLS11, Proposition 3.3.7]): For a lattice N and fan Σ in $N_\mathbb{R}$. We consider $i = \overline{\phi} : N' \hookrightarrow N$ as an inclusion of a finite index sublattice, then Σ can be also considered as a fan in $N'_\mathbb{R}$. Then i induces a toric morphism $\phi : X_{\Sigma,N'} \to X_{\Sigma,N}$. Let $G = N/N'$, show that ϕ exhibits $X_{\Sigma,N}$ as $X_{\Sigma,N'}/G$.

In the later discussions, we need to pay attention to toric varieties without torus factor. We characterize it here. We refer to [CLS11, Proposition 3.3.9] for a full proof.

Theorem 6.4. (Exercise 6.2) *TFAE:*

(1) X_Σ has a torus factor, i,e, there exists an fan Σ' such that $X_\Sigma \simeq X_{\Sigma'} \times \mathbb{T}$.
(2) There is a non-constant toric morphism $X_\Sigma \to \mathbb{C}^\times$.
(3) For all 1-dimensional cones ρ (i.e. rays), their ray generators u_ρ does not span $N_\mathbb{R}$ (Recall that we define u_ρ at the begining of page 5).

Next, we will give an important example that has no torus factor, and non-compact(non-complete) toric variety: Blow up at a point.

Example 6.5. We take $N = \mathbb{Z}^2$ and fan Σ spanned by faces of the cone $\mathbb{R}^2_{\geq 0}$. Then we know that $X_\Sigma = \mathbb{C}^2$.

We consider another fan Σ' described in Example 4.4-(1).

Then we have that $U_{\sigma_1} = \mathrm{Spec}(\mathbb{C}[x, x^{-1}y])$ and $U_{\sigma_2} = \mathrm{Spec}(\mathbb{C}[xy^{-1}, y])$, and they glued along $U_\tau = \mathrm{Spec}(\mathbb{C}[xy, (x^{-1}y)^{\pm 1}])$ $(\tau = \sigma_1 \cap \sigma_2)$. Then we change of coordinates $x = u, v = y, s = xy^{-1}, t = x^{-1}y$, we have that

$$U_{\sigma_1} = \mathrm{Spec}(\mathbb{C}[u, v, t]/\langle tu = v\rangle), \quad U_{\sigma_2} = \mathrm{Spec}(\mathbb{C}[u, v, s]/\langle u = sv\rangle)$$

and they are glued along $st = 1$.

Now, we recall that

$$\mathrm{Bl}_{(0,0)}\, \mathbb{C}^2 = \{([x_0, x_1], u, v) \in \mathbb{P}^1 \times \mathbb{C}^2 : x_0 v = x_1 u\}.$$

We cover it by the affine cover

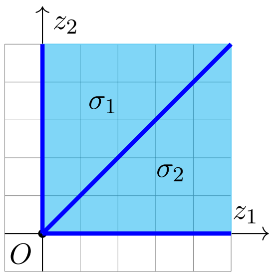
$$U_1 = \{x_0 \neq 0\} \cap \mathrm{Bl}_{(0,0)}\, \mathbb{C}^2 = \{u = x_1/x_0 v\}, \quad U_2 = \{x_1 \neq 0\} \cap \mathrm{Bl}_{(0,0)}\, \mathbb{C}^2 = \{v = x_0/x_1 u\}.$$

Then we change of coordinates $t = x_1/x_0$ and $s = x_0/x_1$ to see that they glued along $st = 1$.

Then it is clear that we have $X_{\Sigma'} = \mathrm{Bl}_{(0,0)}\, \mathbb{C}^2$.

On the other hand, the identity map id_N is compatible with fans Σ' and Σ, which induces an toric morphism. By certain computation, we claim that id_N induces exactly the blow up map

$$X_{\Sigma'} \to X_\Sigma, \quad ([x_0, x_1], u, v) \mapsto (u, v).$$



Notice that for $\sigma = \mathbb{R}^2_{\geq 0}$, we have $p_\sigma = (0, 0)$ (Recall the definition of p_σ in Section 5). Then we generalize the example in the follow way:

For any fan Σ and a smooth cone $\sigma = \mathrm{Cone}(u_1, \cdots, u_n)$ in Σ (recall the definition of smooth cone in Theorem 3.10), we set $u_0 = u_1 + \cdots + u_n$. Then we can define another fan

$$\Sigma^*(\sigma) = \Sigma \setminus \{\sigma\} \cup \{\mathrm{Cone}(\{u_0, u_1, \cdots, u_n\} \setminus \{u_i\}), i = 1, 2, \cdots, n\}.$$

Proposition 6.6. (Exercise 6.3) *The identity map* id_N *induces the blow up at* p_σ*:*

$$X_{\Sigma^*(\sigma)} = \mathrm{Bl}_{p_\sigma}(X_\Sigma) \to X_\Sigma.$$

Hint: Actually, the operation is essential local. So, still only need to verify this property for the $X_{[0,\infty)^n} = \mathbb{C}^n$ case.

Here, we still require that σ is a smooth cone to blow up. In general, we can study further refinement (for example a more delicate version of subdivision is presented in [CLS11, Section 11.1]) to resolve singularities.

Definition 6.7. For a proper morphism of two varieties $\phi : X' \to X$, we say it is a resolution of singularities if X' is smooth and irreducible, and ϕ induces an isomorphism $\phi : \phi^{-1}(X_{\mathrm{reg}}) \xrightarrow{\simeq} X_{\mathrm{reg}}$, where X_{reg} is the regular locus.

For a toric morphism, we say it is a toric resolution (of singularities) if it is a resolution of singularities. It is not surprise that we can use certain combinatorial operation on fans to cook up resolutions.

Definition 6.8. We say a fan Σ' is a refinement of Σ if $\Sigma'(1) \subset \Sigma(1)$ and each cone of Σ' is contained in some cone of Σ.

It is clear that a refinement (together with id_N) induces a toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$. We have the following result about existence.

(Exercise 6.4) Recall the conifold singularity in Example 3.11-(3). Try to construct Atiyah flop via toric resolutions. Hint: Example 7.6.4 of [CLS11]

Proposition 6.9 ([CLS11, Theorem 11.1.9]). *Every fan Σ has a refinement Σ' such that Σ' is smooth and $\phi : X_{\Sigma'} \to X_\Sigma$ is a toric resolution with projective fiber.*

7. Weil divisors on toric variety

We recall basic notion of divisors here.(Remember that we always assume normality.)

For $D \subset X$ a closed irreducible subvariety of codimension 1, we say it is a prime (Weil) divisor.

Definition 7.1. We define

$$\mathrm{Div}(X) = \bigoplus_{D \text{ prime}} \mathbb{Z}D,$$

the free abelian group generated by prime divisors. Elements there in are called Weil divisors.

We take $D = \sum a_i D_i \in \mathrm{Div}(X)$. We say $D \geq 0$ if $a_i \geq 0$ for all d_i.

We set $k(X)$ the field of rational functions on X. For any prime divisor and $f \in k(X)^\times$, we can define an integer called the vanishing order of f along D, which actually form a ring homomorphism $\mathrm{ord}_D : k(X)^\times \to \mathbb{Z}$. Then for $f \in k(X)^\times$, we can define

$$\mathrm{div}(f) = \sum_D : \mathrm{ord}_D(f)D.$$

This is in fact a finite sum since a non-zero ration function only has zero/pole along finitely many prime divisors. Then the set of *principal divisors*

$$\mathrm{PDiv}(X) = \{\mathrm{div}(f) : f \in k(X)^\times\}$$

form a subgroup of $\mathrm{Div}(X)$.

Definition 7.2. We consider the divisor class group (or $(n-1)$-order Chow group) as

$$\mathrm{Cl}(X) = A_{n-1}(X) = \mathrm{Div}(X)/\,\mathrm{PDiv}(X).$$

For any Weil divisor D, we can define a sheaf of $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ in the following way. We set $k(X)$ the field of rational functions on X. Then we set

$$\mathcal{O}_X(D)(U) \coloneqq \{0\} \cup \{f \in k(X)^\times : (\mathrm{div}(f) + D)|_U \geq 0\}.$$

It is clear that $\mathcal{O}_X(D)$ is a $\mathcal{O}_X$-modules sheaf: It is a $\mathcal{O}_X$-modules (pre-sheaf) since regular function has no poles. Next, if f satisfy a condition on poles and zeros locally, then f must satisfy the condition globally.

Now we go to toric varieties. We consider a particular class of prime divisors. By the OCC Theorem 5.3, we have that for any rays $\rho \in \Sigma(1)$, we have a $n-1$-dimension orbit $O(\rho)$. Then it defines a prime divisor:

$$D_\rho \coloneqq \overline{O(\rho)} = \bigcup_{\rho \prec \sigma} O(\sigma), \tag{7.1}$$

which is indeed a prime divisor by Theorem 5.3. They comes from $\mathbb{T}$-orbits, so they are $\mathbb{T}$-invariant divisors. Then we set

$$\mathrm{Div}_{\mathbb{T}}(X_\Sigma) \coloneqq \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho \simeq \mathbb{Z}^{\Sigma(1)}.$$

We know that Laurent polynomials define rational functions on X_Σ, then we can compute their principal divisors.

Lemma 7.3. *For $m \in M$, we have $\mathrm{ord}_{D_\rho}(\chi_m) = \langle m, u_\rho\rangle$ where u_ρ is the ray generator of ρ.*

Proof. u_ρ is the ray generator of ρ, then we can extend it to a $\mathbb{Z}$-basis of N. Therefore, we may assume $u_\rho = e_1$ and $\rho = \mathbb{R}_{\geq 0}e_1$. In the affine tori U_ρ, we first know that $U_\rho = \mathrm{Spec}(\mathbb{C}[x_1, x_2^{\pm 1}, \cdots, x_n^{\pm 1}])$. Then $D_\rho|_{U_\rho} = \{x_1 = 0\}$.

Now, by the formula of $\chi_m(x) = x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$, where $m = (m_1, \cdots, m_n)$. Then it is clear that the vanishing order of $\chi_m(x)$ alone $\{x_1 = 0\}$ is $m_1 = \langle m, u_\rho\rangle$. ☐

Corollary 7.4. *For $m \in M$, we have $\mathrm{div}(\chi_m) = \sum_\rho \langle m, u_\rho\rangle D_\rho$.*

We have the following theorem for computing $A_{n-1}(X_\Sigma)$.

Theorem 7.5 ([CLS11, Theorem 4.1.3])**.** *We have the short exact sequence*

$$M \to \mathrm{Div}_{\mathbb{T}}(X_\Sigma) \to A_{n-1}(X_\Sigma) \to 0.$$

The first map is $m \mapsto \mathrm{div}(\chi_m)$, and the second map is $\mathrm{Div}_{\mathbb{T}}(X_\Sigma) \subset \mathrm{Div}(X_\Sigma) \twoheadrightarrow A_{n-1}(X_\Sigma)$.

(Exercise 7.1)*: Moreover X_Σ has no torus factor if and only if $M \to \mathrm{Div}_{\mathbb{T}}(X_\Sigma)$ is actually injective. (So the above short exact sequence can have the term $0 \to M$ from the left).*

The main point of this theorem is that: for any Weil divisor, it is rational equivalent (i.e. the relation mod principal divisors) to a $\mathbb{T}$-invariant divisor; and such a $\mathbb{T}$-invariant representation of divisor classes is unique upto $\mathbb{T}$-invariant principal divisors.

Proof. Hard part: $\mathrm{Div}_{\mathbb{T}}(X_\Sigma) \to A_{n-1}(X_\Sigma)$ is surjective. It means that for any divisor class, there exists a $\mathbb{T}$-invariant representative in the class. It involves general discussion on divisors, we omit this part.

It is clear that the composition $M \to \mathrm{Div}_{\mathbb{T}}(X_\Sigma) \to A_{n-1}(X_\Sigma)$ is 0. We only need to show that if $[D] = 0$ for $D \in \mathrm{Div}_{\mathbb{T}}(X_\Sigma)$, we have $D = \mathrm{div}(\chi_m)$ for some $m \in M$.

Because $[D] = 0$, it means that there exists a random rational function f (not necessarily $\mathbb{T}$-invariant) such that $D = \mathrm{div}(f)$. Remember that X_Σ and $\mathbb{T}$ are birational equivalent, then $f \in \mathbb{C}(M) = Frac(\mathbb{C}[M])$. On the other hand, $\mathrm{div}(f) = D = \sum_\rho a_\rho D_\rho$ is $\mathbb{T}$-invariant, then we have $\mathrm{div}(f|_{\mathbb{T}}) = \mathrm{div}(f)|_{\mathbb{T}} = D = \sum_\rho a_\rho D_\rho|_{\mathbb{T}} = 0$ (since $D_\rho \cap \mathbb{T} = \varnothing$ by OCC). Therefore, it means that $f|_{\mathbb{T}}$ is actually a regular function on $\mathbb{T}$. We can apply the same discussion to $-D$, which proves that $f^{-1}|_{\mathbb{T}}$ is also a regular function on $\mathbb{T}$. Therefore, we have $f|_{\mathbb{T}}$ is a non-zero regular function on $\mathbb{T}$. In other word, we have that $f \in \mathbb{C}[M] \setminus \{0\}$.

It means that $f = c\chi_m$ for $c \neq 0$, and then we have $D = \mathrm{div}(c\chi_m) = \mathrm{div}(\chi_m)$. □

Corollary 7.6. *For toric varieties, $A_{n-1}(X_\Sigma)$ is finitely generated.*

Example 7.7. (1) For $\mathbb{P}^n$, we have that $\Sigma(1) = \{e_i : i = 0, ..., n\}$ where $e_0 = -e_1 - \cdots - e_n$, and e_i standard vectors for $i \geq 1$. Then for each i, $D_i = D_{\rho_i} = \{x_i = 0\}$ for the homogeneous coordinate $[x_0, \cdots, x_n]$.

Then the map $M \to \mathrm{Div}_{\mathbb{T}}(X_\Sigma)$ is given by

$$m \mapsto (-m_1 - \cdots - m_n, m_1, \cdots m_n),$$

which is rank n. Moreover, $\mathbb{P}^n$ has no torus factor. Then we have that

$$A_{n-1}(\mathbb{P}^n) = \mathrm{coker}(\mathbb{Z}^n \to \mathbb{Z}^{n+1)} \simeq \mathbb{Z}.$$

Here, we see that all D_i are equivalent to the divisor H given by an arbitrary hyperplane.

(2) For the Hirzebruch surface H_r (recall Exercise 4.2): we have

$$\Sigma(1) = \{(-1, r), (0, 1), (1, 0), (0, -1).\}$$

The linear map $M \to \mathrm{Div}_{\mathbb{T}}(X_\Sigma)$ is in fact given by the matrix

$$\begin{bmatrix} -1 & r \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then we know that $A_1(H_r) = \mathbb{Z}^2$.

In fact, it is a general observation that when we identify $M = \mathbb{Z}^n$, then the map $M \to \mathrm{Div}_{\mathbb{T}}(X_\Sigma)$ is given by the matrix

$$\begin{bmatrix} u_{\rho_1}^t & \cdots & u_{\rho_r}^t \end{bmatrix}^t.$$

It enables you to compute $A_{n-1}(X_\Sigma)$ using smith normal form.

Lastly, we want to study the global section of the sheaf $\mathcal{O}_{X_\Sigma}(D)$ for $D \in \mathrm{Div}_{\mathbb{T}}(X_\Sigma)$.

Proposition 7.8. *For $D \in \mathrm{Div}_{\mathbb{T}}(X_\Sigma)$, we have*

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\mathrm{div}(\chi_m)+D\geq 0} \mathbb{C}\chi_m \subset \mathbb{C}[M].$$

Proof. Recall in Proposition 2.3, we see that $k(X_\Sigma) = \mathbb{C}(M)$. As the proposition predicted, we first prove that $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \subset \mathbb{C}[M] \subset \mathbb{C}(M)$, this means that we need to show that, for $f \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$, $f|_{\mathbb{T}}$ is actually a regular function on $\mathbb{T}$. The argument is basically the same as Theorem 7.5. We repeat it here.

To see this, by definition of $\mathcal{O}_{X_\Sigma}(D)$, we have $(\mathrm{div}(f) + D)|_{\mathbb{T}} \geq 0$ since $\mathbb{T}$ is an open set of X_Σ. On the other hand, by OCC, we see that $\mathbb{T} = U_{\{0\}}$ and for all ray ρ, $D_\rho \cap U_{\{0\}} = \varnothing$. So, the condition $(\mathrm{div}(f) + D)|_{\mathbb{T}} \geq 0$ exactly means that $\mathrm{div}(f)|_{\mathbb{T}} = \mathrm{div}(f|_{\mathbb{T}}) \geq 0$. This means that $f|_{\mathbb{T}}$ has no poles on $\mathbb{T}$, i.e. $f \in \mathbb{C}[M]$. Afterward, we pick $m \in M$ such that $f = \chi_m$.

Since $\mathbb{T}$ acts on X_Σ and D is a $\mathbb{T}$-invariant divisor, we have $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ is a $\mathbb{T}$-invariant subspace of $\mathbb{C}[M]$. Then we use Proposition 1.4 to see that $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ is a direct sum of $\mathbb{C}\chi_m$ for $\chi_m \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$. Then by definition, this means that $\mathrm{div}(\chi_m) + D \geq 0$. □

Graphical representation: For $D = \sum_\rho a_\rho D_\rho \in \mathrm{Div}_{\mathbb{T}}(X_\Sigma)$, we have

$$\mathrm{div}(\chi_m) + D \geq 0 \iff \langle m, u_\rho\rangle \geq -a_\rho, \forall \rho \in \Sigma(1).$$

Then we set

$$P_D = \{m \in M_{\mathbb{R}} : \langle m, u_\rho\rangle \geq -a_\rho, \forall \rho \in \Sigma(1)\}.$$

Tautologically, we have

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m\in P_D\cap M} \mathbb{C}\chi_m.$$

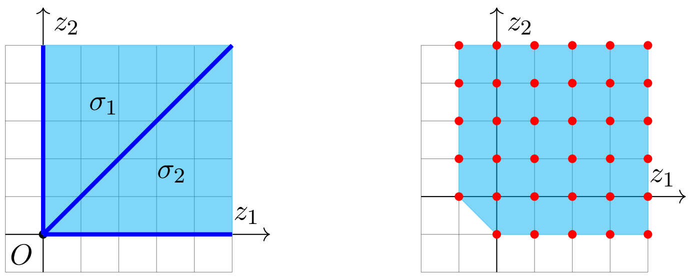
For this P_D, we notice the following operation

$$P_{kD} = kP_D, \quad P_{D+\mathrm{div}(\chi_m)} = P_D - m, \quad P_{D+E} \subset P_D + P_E.$$

Example 7.9. (1) We first consider the blow-up example. The fan is given here. Now, we pick the divisor $D = D_0 + D_1 + D_2$ for 3 rays, where $u_0 = e_1 + e_2, u_1 = e_1, u_2 = e_2$. Then we have

$$P_D = \{(m_1, m_2) : m_1 + m_2 \geq -1,\ m_1 \geq -1,\ m_2 \geq -1\}.$$

By the picture, we know that $\Gamma(\mathrm{Bl}_0(\mathbb{C}^2), \mathcal{O}_{\mathrm{Bl}_0(\mathbb{C}^2)}(D))$ is infinite dimensional.



LHS=The fan; RHS=P_D.

(2) For the Hirzebruch surface H_r (recall Exercise 4.2): we have 4 ray generators

$$\{(-1, r), (0, 1), (1, 0), (0, -1)\} = \{u_1, u_2, u_3, u_4\}.$$

We take $D_a = aD_1 + D_2$ for $a \in \mathbb{Z}$. Then we have

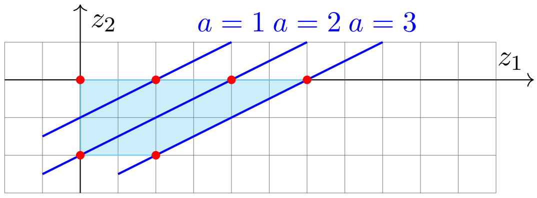
$$P_{D_a} = \{(m_1, m_2) : rm_2 - m_1 \geq -a, m_2 \geq -1, m_1 \geq 0, m_2 \leq 0\}.$$

We look at an $r = 2$ example. Then on picture, we have. Here P_{D_a} is the area inside the cyan area and above the slanted lines with given a.

Therefore, we have

$$\dim \Gamma(H_2, \mathcal{O}_{H_2}(D_1)) = 2, \quad \dim \Gamma(H_2, \mathcal{O}_{H_2}(D_2)) = 4, \quad , \dim \Gamma(H_2, \mathcal{O}_{H_2}(D_3)) = 6.$$

Proposition 7.10 (Exercise 7.2)**.** *For a compact toric variety X_Σ (i.e. $|\Sigma| = N_{\mathbb{R}}$), we have*



RHS=P_{D_a} for $a = 1, 2, 3$.

(1) $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}) = \mathbb{C}$.
(2) For all $D \in \mathrm{Div}_{\mathbb{T}}(X_\Sigma)$*,* P_D *is a polytope (i.e. a bounded polyhedron).*
(3) $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ *is finite dimensional for all Weil divisors* D *(not just for invairant divisors).*

(Notice: (1), (3) can follow from general (but hardcore) theorems in algebraic/complex geometry. But here, the point is we can give an elementary proof for toric varieties without anything difficult.)

8. Line bundles and Cartier divisors on toric variety

Again, let us start from review.

Let $\mathrm{Pic}(X)$ the group of line bundles on X, then we have $\mathrm{Pic}(X) \hookrightarrow A_{n-1}(X)$ (in fact, it need normality). The morphism is surjective if $\forall x \in X$, local rings $\mathcal{O}_{X,x}$ are UFD (for example when X is smooth). For a Weil divisor D, we say it is Cartier if its divisor class $[D] \in A_{n-1}(X)$ is represented by a line bundle. It just means that for an open cover U_i of X, $D|_{U_i}$ is a principal divisor. This is true since line bundles have transition functions and $D|_{U_i}$ is the principal divisor associated with the transition function. We denote the subgroup of Cartier divisors as $\mathrm{CDiv}(X_\Sigma)$. Clear, principal divisors are Cartier, and they defines the trivial line bundles. Then we have $\mathrm{Pic}(X) \simeq \mathrm{CDiv}(X)/\mathrm{PDiv}(X)$. Tautologically, we have, for a Cartier divisor D, the sheaf $\mathcal{O}_X(D)$ is an rank 1 locally free sheaf, which represent the section sheaf of the line bundle corresponds to $[D]$.

In the toric case X_Σ, we denote the subgroup $\mathrm{CDiv}_{\mathbb{T}}(X_\Sigma)$ of $\mathrm{Div}(X_\Sigma)$ consists of $\mathbb{T}$-invariant Cartier divisors. Then by definition, we automatically have the theorem, which follows from Theorem 7.5,

Theorem 8.1. *We have the short exact sequence*

$$M \to \mathrm{CDiv}_{\mathbb{T}}(X_\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0.$$

When X_Σ has no torus factor, we have that $M \to \mathrm{CDiv}_{\mathbb{T}}(X_\Sigma)$ is injective.

Now, we gonna describe $\mathrm{CDiv}_{\mathbb{T}}(X_\Sigma)$ in a more precise using our fan Σ.

We state the affine result without proof and refer to [CLS11, Proposition 4.2.2].

Proposition 8.2. *For a strongly convex rational cone $\sigma \subset N_{\mathbb{R}}$, we have the map $M \to \mathrm{CDiv}_{\mathbb{T}}(X_\sigma)$ is surjective. I.e. all $\mathbb{T}$-invariant Cartier divisors on affine toric varieties are principal.*

Consequently by Theorem 8.1, there is no non-trivial line bundle on affine toric variety.

Example 8.3. Consider the conifold singularity X_σ in Example 3.11-(3). The result implies that $\mathrm{Pic}(X_\sigma) = 0$. But you can compute by Theorem 7.5 that we have $A_2(X_\sigma) = \mathbb{Z}$. In fact, for D_i corresponding 4 ray generators of X_σ, they are not Cartier, and $\sum_i a_i D_i$ is Cartier if and only if $\sum_i a_i = 0$.

We the help of affine result, we describe $\mathrm{CDiv}_{\mathbb{T}}(X_\sigma)$ for general toric varieties. Here, we set $\Sigma_{max} \subset \Sigma$ the set of maximal cones: cones that is not a proper subset of any other cones.

Theorem 8.4. *For a toric variety X_Σ and $D \in \mathrm{Div}_{\mathbb{T}}(X_\sigma)$, then TFAE:*
(1) D is Cartier,
(2) D is principal on toric affines U_σ for all $\sigma \in \Sigma$.
(3) For each $\sigma \in \Sigma$, there exists $m_\sigma \in M$ such that for all $\rho \in \Sigma(1)$ and $\rho \subset \sigma$ we have $\langle m_\sigma, u_\rho \rangle = -a_\rho$.
(4) For each $\sigma \in \Sigma_{max}$, there exists $m_\sigma \in M$ such that for all $\rho \in \Sigma(1)$ we have $\langle m_\sigma, u_\rho \rangle = -a_\rho$.
With m_σ given for the Cartier divisor D, we have: a) m_σ is unique modulo $M(\sigma) := \sigma^\perp \cap M$; b) τ is a face of σ, then $m_\sigma \equiv m_\tau \mod M(\tau)$.

Proof. (1) $\iff$ (2) from the affine result Theorem 8.1. (2) $\Rightarrow$ (3): Since $D|_{U_\sigma}$ principal, we have $D|_{U_\sigma} = \chi_{-m_\sigma}$ for some m_σ[7]. Then since $\chi_{-m_\sigma} = D|_{U_\sigma} = \sum_\rho a_\rho D_\rho$ for $\rho \in \Sigma(1)$ and $\rho \subset \sigma$, we have $\mathrm{ord}_{D_\rho}(\chi_{-m_\sigma}) = \langle -m_\sigma, u_\rho \rangle = a_\rho$ by Lemma 7.3. (3) $\Rightarrow$ (2): By the condition, we have $D = \sum_\rho \langle -m_\rho, u_\rho \rangle D_\rho$. Then we have $D|_{U_\sigma} = \sum_{\rho \subset \sigma} \langle -m_\rho, u_\rho \rangle D_\rho = \sum_{\rho \subset \sigma} \langle -m_\sigma, u_\rho \rangle D_\rho = \chi_{-m_\sigma}$. (3) $\Rightarrow$ (4) trivial. (4) $\Rightarrow$ (3) is true since if m_σ works for σ, then m_σ should works for all its faces.

The uniqueness of m_σ modulo $M(\sigma)$ follows directly from its definition. The compatibility (b) comes from the uniqueness since m_σ can be choicen as m_τ when τ is a face of σ. $\square$

[7] A minus here for some conventional reason.

Therefore, for a Cartier divisor D, we have $(U_\sigma, \chi_{-m_\sigma})$ form a gluing data that gives us a line bundle on X_Σ. The compatibility condition shows that, if we think Σ as a direct poset w.r.t 'being face' relation, we have a natural isomorphism

$$\mathrm{CDiv}_{\mathbb{T}}(X_\sigma) = \varprojlim_{\Sigma} M/M(\sigma).$$

Next, we consider the differential sheaf and canonical divisor. To simplify discussion, we only consider X_Σ smooth case. Then we have the Euler sequence:

Theorem 8.5 ([CLS11, Theorem 8.1.6]). *For a smooth toric variety X_Σ without torus factor, we have an short exact sequence of $\mathcal{O}_{X_\Sigma}$-module sheaves*

$$0 \to \Omega^1_{X_\Sigma} \to \oplus_{\rho\in\Sigma(1)} \mathcal{O}_{X_\Sigma}(-D_\rho) \to A_{n-1}(X_\Sigma) \otimes \mathcal{O}_{X_\Sigma} \to 0.$$

We skip the proof since I am not sure if you know enough algebraic geometry to describe $\Omega^1_{X_\Sigma}$.

Example 8.6. For $\mathbb{P}^n$, we have the usual Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

We use it to compute the canonical divisor. In the smooth case, we have $\omega_{X_\Sigma} = \wedge^n \Omega^1_{X_\Sigma}$.

Corollary 8.7 (Exercise 8.1)**.** *For a smooth toric variety X_Σ without torus factor, we have*

$$\omega_{X_\Sigma} = \mathcal{O}_{X_\Sigma}(-\sum_{\rho\in\Sigma(1)} D_\rho).$$

Or equivalently, we have

$$[K_{X_\Sigma}] = [-\sum_{\rho\in\Sigma(1)} D_\rho].$$

In particular, it implies that toric varieties equip with the toric boundary $(X_\Sigma, \sum_{\rho\in\Sigma(1)} D_\rho)$ is log Calabi-Yau (simply say, it means that (Y, D) is a log pair for a Y smooth and D is a divisor on it such that $[K_Y + D] = 0$.)

Remark 8.8. The corollary is also true if X_Σ has torus factor and singular as stated in [CLS11, Theorem 8.2.3]. But one should be careful on the definition of canonical divisor: On normal variety, singularities show up in codimension 2. So, we can first define the canonical divisor K_U on the smooth locus $X_{reg} = U(\subset X)$ as a Cartier divisor (of its canonical line bundle ω_U). Then we extend by 0 using normality. This gives us a correct notion of canonical divisor $K_X = j_{U*}K_U$, but it would merely be a Weil divisor!

In this case, the statement of the theorem is that the Weil divisor K_{X_Σ} for a toric variety (which is always normal) is the Weil divisor $-\sum_{\rho\in\Sigma(1)} D_\rho$.

Example 8.9 (Exercise 8.2)**.** We say a smooth variety X is Calabi-Yau (in a very loose sense) if ω_X is trivial. Show that a toric X_Σ is Calabi-Yau if and only if there exists $m \in M$ such that $\langle m, u_\rho\rangle = 1$ for all $\rho \in \Sigma(1)$. Consequently, show that toric Calabi-Yau in this sense must be non-compact.

9. Projective toric variety

So far, we only take intrinsic view of toric variety. In this section and the next section, we develop two extrinsic points of view related with each other. The main concern in this section is one can find very ample line bundles from lattice polytope.

We start from general construction for (equivariant) rational maps from toric varieties into projective spaces. For a $\mathbb{T}$-invariant Cartier divisor $D = \sum_\rho a_\rho D_\rho$, then by Proposition 7.8, we have $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C}\chi_m$ for a polytope P_D. Here, we set $P_D \cap M = \{m_0, \cdots, m_N\}$ (It is probably that $N = 0$, which means that only constant section there. But let us assume that $N \geq 1$ to avoid the trivial situation.) Then we can define a rational map, which is equivariant (since χ_{m_i} are),

$$f_D : X_\Sigma \dashrightarrow \mathbb{P}(\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))) = \mathbb{P}^N. \tag{9.1}$$

Definition 9.1. For D as above. We say D is basepoint free if f_D is a global defined morphism (rather than rational). We say D is very ample if f_D is a closed embedding. We say D is ample if kD is very ample for some $k \geq 1$.

Remark 9.2. In fact, for all (not necessarily invariant) Cartier divisors, this is a reasonable definition. However, we will only consider toric case we explained here, which force f_D to be equivariant (which is generally not true).

Let $\Delta \subset M_\mathbb{R}$ to be a lattice polytope: it is a convex hull of a finite set $S \subset M$, or equivalently, it is a finite intersection of half-planes that is bounded with integer vertices. We also assume the linear span of Δ to be $M_\mathbb{R}$, in this case, we say Δ is full dimensional. In this lecture, we always assume Δ is full dimensional!

A subset $Q \subset \Delta$ is a called a face if there exists an affine hyperplane $H_{u,b}$ with normal vector u such that $Q = \Delta \cap H_{u,b}$ and $P \subset H^+_{u,b}$ (i.e. the half space of u-side rather then $(-u)$-side). *As a convention*, we also treat Δ itself as a face of Δ. Since faces are all polytope, not necessarily full dimension, then we can also discuss the dimension of a face (we allow not just of co-dimensional 1 faces).

For a full dimensional lattice polytope Δ, we have: For facets F, we can find a couple of vectors $u_F \in N$ and integers a_F such that

$$\Delta = \{m \in M_\mathbb{R} : \langle m, u_F \rangle \geq -a_F, \forall F\}.$$

This is called facets presentation of Δ, and it is clear that u_F are inward-pointing normal vector of the facet F. Use the facets presentation, we can define a fan in the following way: For every face Q of Δ, we define the following cone

$$\sigma_Q = \mathrm{Cone}(u_F : Q \subset F).$$

Then we have the following technical result:

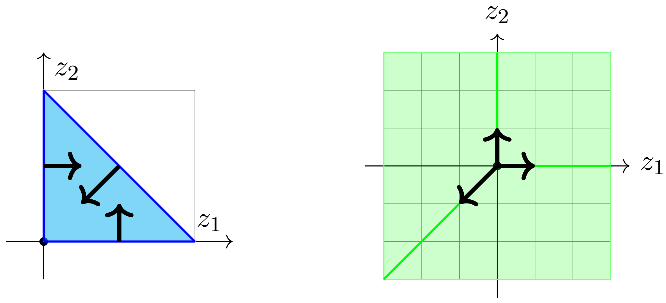
Proposition 9.3. *We set $\Sigma_\Delta = \{\sigma_Q : Q$ is a face of $\Delta\}$. Then Σ_Δ is a complete fan (called the normal fan of Δ), precisely:*

(1) σ_Q is a strongly convex, N-rational, polyhedral cone for each Q;
(2) For all Q, each face of σ_Q is in Σ_Δ;
(3) For $\sigma_Q, \sigma_{Q'} \in \Sigma_\Delta$, $\sigma_Q \cap \sigma_{Q'}$ is a face of both $\sigma_Q, \sigma_{Q'}$.
(4) $|\Sigma_\Delta| = N_\mathbb{R}$.
(5) $\dim Q + \dim \sigma_Q = n$.

Definition 9.4. We can define a toric variety X_Δ as X_{Σ_Δ}. It is a compact (Ex: why?) toric variety.

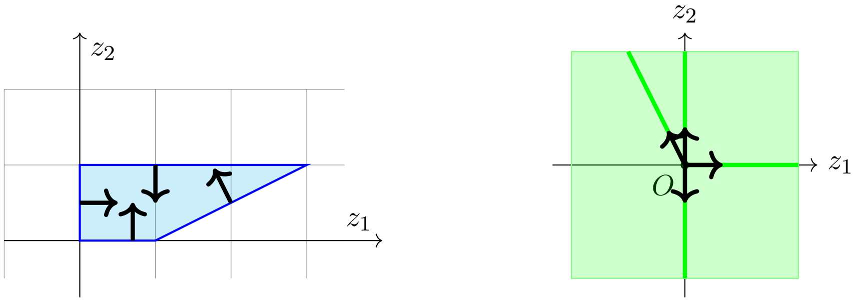
A simple observation is for $k \geq 1$, $\Sigma_\Delta = \Sigma_{k\Delta+m}$. So, we have $X_\Delta = X_{k\Delta+m}$ as an abstract toric variety. However, will see that the choices of k and m affect its projective embedding.

Example 9.5. (1) Consider the (unit) simplex Δ_2 in $M_\mathbb{R}$, its normal fan is the $\mathbb{P}^2$-fan.



Δ_2 Σ_{Δ_2}

Δ_2 for $\mathbb{P}^2$ and its normal fan.



Δ for H_2 and its normal fan.

(2) Consider the polytope described below, its normal fan is the H_2-fan.
(3) For the polytope Δ, the Orbit-Cone correspondence can be formulate as Orbit-faces correspondence. In fact, faces Q, cones σ_Q, and orbits $O(\sigma_Q)$ are 1-1 correspondent, and $\dim O(\sigma_Q) = \dim Q$. In particular, vertices of Δ correspondence to $\mathbb{T}$ fixed points.
(4) For a smooth maximal dimension cone σ_v, which correspondence a vertex v, and a $\mathbb{T}$ fixed point $p_{\sigma_v} = p_v$. We can study its blow up $Bl_{p_v}(X_\Delta)$ in Proposition 6.6. Here, we notice that a σ_v is smooth is equivalent to the corner around v form a basis. Near such a vertex, if we remove the corner, the resulting polytope gives the fan $\Sigma^*(\sigma_v)$. This gives a polytope explanation of the toric blow up constriction.

So far, we are still working intrinsically. Now, we consider additional information from Δ. We first notice that the facet normal vectors u_F are actually ray generators of the normal fan Σ_Δ: $\Sigma_\Delta(1) = \{\mathbb{R}_{\geq 0} u_F\}$. Then we can define a Weil divisor using integers a_F

$$D_\Delta = \sum_F a_F D_F.$$

Proposition 9.6. *D_Δ is a Cartier divisor of X_Δ and $[D_\Delta] \neq 0$.*

Proof. By Theorem 8.4, we need to specify m_{σ_Q} for $\sigma_Q \in (\Sigma_\Delta)_{max}$. We can check that $(\Sigma_\Delta)_{max} = \{\sigma_v : \text{vertices of } \Delta\}$. Then we define $m_{\sigma_v} = v$, this is a reasonable choice since for any $v \in F$, we have $\langle v, u_F \rangle = -a_F$ by the definition of facet presentation of Δ! By the construction of Theorem 8.4, it is clear the choices of $m_{\sigma_v} = v$ gives the divisor $D_\Delta = \sum_F a_F D_F$.

The proof for D_Δ is not principal will be left as an Exercise 9.1 (Hint: use Proposition 7.8, Proposition 7.10). ☐

Now, we consider section computation of D_Δ. Recall again that for every $\mathbb{T}$-invariant Cartier divisor D, we can define a polytope P_D such that $\Gamma(X_\Delta, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C}\chi_m$ by Proposition 7.8. We make the following observation.

Lemma 9.7. *For a lattice polytope Δ and its associated Cartier divisor D_Δ, we have $P_{D_\Delta} = \Delta$.*

Therefore, we have

$$\Gamma(X_\Delta, \mathcal{O}_{X_\Delta}(D_\Delta)) = \bigoplus_{m\in\Delta\cap M} \mathbb{C}\chi_m.$$

Remark 9.8. For a Laurent polynomial $f = \sum_m a_m\chi_m \in \mathbb{C}[M]$, we set $N(f) = \mathrm{Conv}\{m : a_m \neq 0\}$. It is called the Newton polytope of f. Then for a generic polynomial in $\Gamma(X_\Delta, \mathcal{O}_{X_\Delta}(D_\Delta))$, its Newton polytope $N(f) = \Delta$.

Because $k\Delta$ is lattice and full dimension, $k\Delta \cap M$ has at least 3 points. We set the number of lattice to be $N+1$ (for $N \geq 2$). Then the dimension of the vector space of sections is also $N+1$. Follow our discussion from the beginning, we have the following rational map

$$f_{D_{k\Delta}} : X_\Delta \dashrightarrow \mathbb{P}(\Gamma(X_\Delta, \mathcal{O}_{X_\Delta}(D_{k\Delta}))) = \mathbb{P}^N, \quad x \mapsto [\chi_{m_0}(x), \cdots, \chi_{m_N}(x)].$$

The main theorem here is

Theorem 9.9 ([CLS11, Proposition 6.1.10])**.** *The divisor D_Δ is basepoint free, which implies that f_{D_Δ} defined in* (9.1) *is an algebraic morphism. The divisor D_Δ is always ample*[8]*: Precisely, for $n \geq 2$, the divisor kD_Δ is very ample for all $k \geq n-1$.*

Therefore, we have that f_{kD_Δ} is a projective embedding for all $k \geq n-1$, and then $X_\Delta = X_{k\Delta}$ is a projective variety. In this case, we have $f^*_{D_{k\Delta}}\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_{X_{k\Delta}}(kD_\Delta)$.

Idea of the proof. We only explain the ampleness condition here. We can reduce to affine cover (but not in a naive way). Then it is proven in *loc. cit.* the very ampleness of kD_Δ can be reduced to an equivalent combinatorial conditions about kD_Δ. Then it is proven in [CLS11, Proposition 2.2.19] that the combinatorial conditions is true for kD_Δ for all $k \geq n-1$. □

Now, we give examples to illustrate.

Example 9.10. We only consider the $\mathbb{P}^2$ case. We write $x = [x_0, x_1, x_2]$ and $D_0 = \{x_0 = 0\}, D_1 = \{x_1 = 0\}, D_2 = \{x_2 = 0\}$. We will consider three polytopes. One the unit simplex Δ_2 and one the convex hull spanned by $(-1,-1), (-1,3), (3,-1)$, say Δ_1, and $2\Delta_2$.

For Δ_2, its facet presentation gives only one non-zero a_F for the slanted line $-m_1 - m_2 \geq -1$. So, $D_{\Delta_2} = D_0$. And $\Gamma(\mathbb{P}^2, D_{\Delta_2}) = \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y$, where $x = x_1/x_0$ and $y = x_2/x_0$. We naturally identify them with $\Gamma(\mathbb{P}^2, D_{\Delta_2}) = \mathbb{C}x_0 \oplus \mathbb{C}x_1 \oplus \mathbb{C}x_2$ using homogeneous coordinates. Then it is clear that $f_{D_{\Delta_2}}$ is the identity map.

For Δ_1, its facet presentation is $m_1 \geq -1$, $m_2 \geq -1$ and $-m_1 - m_2 \geq -2$. Then we have that $D_{\Delta_1} = D_0 + D_1 + D_2 = -K_{\mathbb{P}^2}$ is the anti-canonical divisor. Then we can find that $\Gamma(\mathbb{P}^2, D_{\Delta_2})$ can be identified with $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ that gives you the spaces of homogeneous cubic polynomials, which is of dimension 10. Then

$$f_{D_{\Delta_1}} : \mathbb{P}^2 \to \mathbb{P}^9, \quad [x_0, x_1, x_2] \mapsto [x_0^a x_1^b x_2^c]_{a+b+c=3}.$$

This map is known as the degree 3 Veronese map in 3-variable. It is also true that $f^*_{D_\Delta}\mathcal{O}_{\mathbb{P}^9}(1) = \mathcal{O}_{\mathbb{P}^2}(3)$.

For $2\Delta_2$, the computation is similar since we can see that $D_{2\Delta_2} = 2D_0$, and the embedding is given by quadratic analogy of $f_{D_{\Delta_1}}$, i.e. the degree 2 Veronese map in 3-variable. Precisely, we have

$$f_{D_{\Delta_1}}([x_0, x_1, x_2]) = [x_0x_0, x_1x_1, x_2x_2, x_0x_1, x_0x_2, x_1x_2] = [Y_0, Y_1, Y_2, Y_3, Y_4, Y_5].$$

Then, we can described its image by the defining equations:

$$Y_0Y_1 = Y_3Y_3,\ Y_0Y_5 = Y_4Y_3,\ Y_0Y_2 = Y_4Y_4,\ Y_3Y_5 = Y_4Y_1,\ Y_3Y_2 = Y_4Y_5,\ Y_1Y_2 = Y_5Y_5.$$

Clearly, these are different projective embeddings.

Exercise 9.1: Consider polytope in Example 9.5-(2) for H_2. Write down an projective embedding $f : H_2 \to \mathbb{P}^5$, try to find the defining equations.

[8] The $n = 1$ case is trivial: the only possible 1-dimensional lattice polytope gives $\mathbb{P}^1$.

Remark 9.11. Under analytic topology and assume X_Δ is smooth, a $\mathbb{T}$-equivariant projective embedding gives a symplectic form on X_Δ (pull-back of Fubini-Study) that is equivariant under $(S^1)^n \subset \mathbb{T}$-action. In particular, this action is effective and Hamiltonian. The polytope Δ shows up as the moment map image. Translation and dilation of the moment image (or Δ) corresponds to choices for defining the moment map and rescale the symplectic form.

In general, a $2n$-dimensional symplectic toric manifold is a $2n$-dimensional connected symplectic manifold plus an effective and Hamiltonian $(S^1)^n$ (plus a moment map of the action). So, we have explained here that X_Δ (plus a choice of the moment map) gives a compact symplectic toric manifold.

A more precise relation is the following: We say a polytope Δ is Delzant if there exists a couple of affine transformations such that the resulting polytope is lattice and its normal fan is smooth. We have the following bijection

$$\{\text{Compact symplectic toric manifold}\}/Symp \leftrightarrow \{\text{Delzant polytope } \Delta\}/,\ (X, \omega, (S^1)^n \curvearrowright X, \mu) \mapsto \mu(X).$$

Well-definedness of the map is the content of Atiyah-Guillemin-Sternberg convexity theorem. Here, we have given a (part of) proof of surjectivity (we didn't specify the moment map and we didn't explain the moment map image is the given Δ, but you can supplement by yourself.) Another proof of surjectivity can be given using the quotient construction we will explain in Section 10.

To prove injectivity, we first observe that different choice of moment map gives translation and rescaling of Δ. So we can only consider the case that Δ is lattice. The uniqueness of normal fan gives the uniqueness of abstract variety, and if we fix k that makes $k\Delta$ very ample, a symplectic form is also fixed. This basically proves the uniqueness (Probably something was missing here, but not too far.)

10. Quotient construction and Global homogeneous coordinate

In this section, we describe the quotient construction of toric variety. For a fan Σ in $N_{\mathbb{R}}$, we assume X_Σ has no torus factor (c.f. Theorem 6.4); this is not so necessary, but make certain conveniences. We refer to [CLS11, Chapter 5] as the reference that general cases are also discussed.

Convention: Recall first that $\Sigma(1)$ is the set of rays, u_ρ is the ray generator of $\rho \in \Sigma(1)$, D_ρ is the prime divisor associated with ρ. We set $|\Sigma(1)| = r \geq n$ (the inequality follows from that X_Σ has no torus factor.

We first state our theorem before inducing new notations.

Theorem 10.1. *For a fan Σ in $N_{\mathbb{R}}$, we assume X_Σ has no torus factor, there exists a closed subvariety $Z(\Sigma) \subset \mathbb{C}^r$ and a group $G \subset \mathbb{T}_r$ and a toric morphism $\mathbb{C}^r \setminus Z(\Sigma) \to X_\Sigma$, such that G acts on $\mathbb{C}^r \setminus Z(\Sigma)$, the toric morphism induces a $\mathbb{T}$-equivariant morphism of algebraic stacks*

$$[(\mathbb{C}^r \setminus Z(\Sigma))//G] \to X_\Sigma$$

that exhibits X_Σ as an almost geometric quotient. When X_Σ is orbifold, this is a geometric quotient (which particularly implies that both stack and variety have 1-1 correspondence closed points).

When X_Σ is smooth, the G action on $\mathbb{C}^r \setminus Z(\Sigma)$ is free. Then we have that [9]

$$[(\mathbb{C}^r \setminus Z(\Sigma))//G] \simeq X_\Sigma.$$

We are not going to explain the precise meaning of different quotients here. We only consider the smooth case to simplify our discussion.

Construction: Because X_Σ, Theorem 7.5 shows that we have an exact sequence

$$0 \to M \to \sum_\rho \mathbb{Z}D_\rho = \mathbb{Z}^r \to A_{n-1}(X_\sigma) \to 0.$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$, we have an exact sequence

$$1 \to \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X_\sigma), \mathbb{C}^\times) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^r, \mathbb{C}^\times) \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \to 1.$$

So $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \otimes \mathbb{C}^\times = N_{\mathbb{C}^\times} = \mathbb{T}_N$, and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^r, \mathbb{C}^\times) = (\mathbb{C}^\times)^r$.

We *define* $G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X_\sigma), \mathbb{C}^\times) \subset (\mathbb{C}^\times)^r$ a subgroup. Since $A_{n-1}(X_\sigma)$ is a finitely generated abelian group, we have G is a product of torus and finite abelian group. We have the following direct computation.

Lemma 10.2. *If we pick a basis e_i of M, then we have*

$$G = \{(t_\rho)_\rho \in (\mathbb{C}^\times)^r : \prod_\rho t_\rho^{\langle e_i, u_\rho \rangle} = 1, \forall i\}.$$

The formula is surely true if we replace e_i by all $m \in M$.

Remark 10.3. With the given basis e_i, you can write $M \to \mathbb{Z}^r$ as a $r \times n$ $\mathbb{Z}$-valued matrix. The i-th column of this matrix gives you exponential of the i-th equation in the definition of G.

Next, we consider the ring $\mathbb{C}[x_\rho : \rho \in \Sigma(1)]$. For any cone $\sigma \in \Sigma$, we *define* a monomial

$$x^{\hat{\sigma}} = \prod_{\rho \not\subset \sigma} x_\rho.$$

Then we *define* the irrelevant ideal (also called Stanley–Reisner ideal)

$$B(\Sigma)\,(\text{or } SR(\Sigma)) = \langle x^{\hat{\sigma}} : \sigma \in \Sigma \rangle.$$

Small observation is that if $\tau \prec \sigma$, then $x^{\hat{\sigma}} | x^{\hat{\tau}}$. So, we only need to generate the ideal by $\sigma \in \Sigma_{\max}$. Then we *define* the closed subvariety

$$Z(\Sigma) = \mathrm{Spec}\,(\mathbb{C}[x_\rho]/B(\Sigma)) \subset \mathbb{C}^r.$$

[9]The point for the statement is that the quotient stack is represented by a scheme [Sta, Lemma 80.11.7] under freeness condition, and the scheme is isomorphic to the toric variety X_Σ using the given morphism.

Geometrically, it is a union of coordinate subspaces.

We set $\widetilde{N} = \mathbb{Z}^r$. For each $\sigma \in \Sigma$, we can define a cone $\widetilde{\sigma} = \mathrm{Cone}(e_\rho : \rho \subset \sigma)$ in $\widetilde{N}_\mathbb{R}$, and $\widetilde{\Sigma} = \{\tau : \forall \tau \prec \widetilde{\sigma}, \forall \sigma \in \Sigma\}$ a fan in $\widetilde{N}_\mathbb{R}$

Lemma 10.4. *We the notation above, $\mathbb{C}^r \setminus Z(\Sigma)$ is a toric variety with fan $\widetilde{\Sigma}$. The toric morphism $\mathbb{C}^r \setminus Z(\Sigma) \to X_\Sigma$ is given by the $\mathbb{Z}$-linear map*

$$\widetilde{N} = \mathbb{Z}^r \to N, \quad e_\rho \mapsto u_\rho.$$

Remark 10.5. Computational remarks. In fact, recall that we know there is a $\mathbb{Z}$-linear map $M \to \mathbb{Z}^r$, then the above is the dual/transport of the matrix.

Now, we present all constructions show up in the theorem. Let's give some example to illustrate the theorem.

Example 10.6. (1) In Example 3.11-(2), we explain the construction of A_1-singularity by hand. Here, we present it (and its small generalization) using the quotient construction: We consider the cone $\sigma = \mathrm{Cone}((0,1),(d,-1))$. Its fan is $\Sigma = \sigma, \mathbb{R}_{\geq 0}(0,1), \mathbb{R}_{\geq 0}(d,-1), \{0\}$.

These two vectors also give the ray generators (so $r = 2$), and σ is the only one maximal cone.

Then we have $B(\Sigma) = \{1\}$ is the zero ideal. So, $Z(\Sigma) = \varnothing$. For the group G, it is given by equations $t_1^{-1}t_2 = 1$ and $t_1^d = 1$. So, we have that $G = \{(\zeta,\zeta) : \zeta^5 = 1\} \simeq \mathbb{Z}/d$. Therefore, the quotient construction gives the geometric quotient

$$[\mathbb{C}^2//(\mathbb{Z}/d)] \to \mathbb{C}^2/(\mathbb{Z}/d).$$

(2) In Example 4.8-(4), we explain the construction of weight projective space as a toric variety. Here, we use the quotient construction to see it is indeed the weight projective space.

Recall the construction, let $(q_0, q_1, \cdots, q_n) \in \mathbb{N}^{n+1}$ with $\gcd(q_0, q_1, \cdots, q_n) = 1$. Consider $N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0, q_1, \cdots, q_n)$. We set u_i the natural projection of $e_i \in \mathbb{Z}^{n+1}$ onto N, and

$$\Sigma = \{\text{Proper subsets of } \{u_0, u_1, \cdots, u_n\}\}.$$

Here, it is easy to see that $r = n+1$. The irrelevant ideal is given by $B(\Sigma) = \langle x_0, x_1, \cdots, x_n\rangle$. So $Z(\Sigma) = \{0\}$. With the lemma, $\mathbb{Z}^{n+1}$ should be understood as the fan of the toric variety $\mathbb{C}^r \setminus Z(\Sigma) = \mathbb{C}^{n+1} \setminus \{0\}$. It remains to compute G, which is given by the equations

$$t_0^{-q_0} t_1 = \cdots = t_0^{-q_n} t_n = 1.$$

I.e. $G = \{(t^{q_0}, \cdots, t^{q_n}) : t \in \mathbb{C}^\times\} \simeq \mathbb{C}^\times$. It is clear that

$$[\mathbb{C}^{n+1} \setminus \{0\}//G] \to \mathbb{P}^n(q_0, q_1, \cdots, q_n)$$

is the weight projective space as both stack and variety.

When $(q_0, q_1, \cdots, q_n) = (1, 1, \cdots, 1)$, the action is free and the quotient stack is presented by the projective space.

(3) Exercise 10.1 Write down the quotient constructions of $\mathrm{Bl}_0(\mathbb{C}^n)$ and the conifold singularity Example 3.11-(3).

Lastly, we discuss the global homogeneous coordinate and the homogenization process. The role of it is similar to the role of usual homogeneous coordinate to projective space.

In the construction before, we define the ring $S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$. We call it the total coordinate ring, or the Cox ring of X_Σ. S is naturally $A_{n-1}(X_\Sigma)$-graded: for monomial $x^\alpha = \prod_\rho x_\rho^\alpha$, we define $\deg(x^\alpha) = [\sum \alpha_\rho D_\rho] \in A_{n-1}(X_\Sigma)$; we set S_α the subspace of degree α homogeneous polynomials.

We will use it for the following two reasons:

- **Homogenization**: For a Weil divisor, we first recall that, we define a polyhedron P_D for every $\mathbb{T}$-invariant Weil divisor $D \in \mathrm{Div}_\mathbb{T}(X_\Sigma)$:

$$P_D = \{m \in M_\mathbb{R} : \langle m, u_\rho\rangle \geq -a_\rho, \forall \rho \in \Sigma(1)\}.$$

We can then define the D-Homogenization of Laurent polynomial $\chi_m = \prod_i t_i^{m_i}$ as

$$x^{\langle m,D\rangle} = \prod_\rho x_\rho^{\langle m,u_\rho\rangle + a_\rho} \in S_{[D]}.$$

In fact, it is direct to verify that the map defines an $\mathbb{C}$-linear isomorphism

$$\chi_m \mapsto x^{\langle m,D\rangle}, \quad \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \xrightarrow{\simeq} S_{[D]}.$$

An advantage to use homogenization here is that: Laurent polynomial gives formula of sections of $\mathcal{O}_{X_\Sigma}(D)$ restrict to $\mathbb{T} \subset X_\Sigma$, but it is implicit "compactified" to X_Σ, while the homogenization gives formula of sections on whole $\mathcal{O}_{X_\Sigma}(D)$ using the global homogeneous coordinate that is global defined.

• **The toric Ideal-Variety correspondence** ([CLS11, Proposition 5.2.7]): Let X_Σ be a orbifold toric variety. Then we have a bijection

$$\{\text{Closed subvarieties of } X_\Sigma\} \leftrightarrow \{\text{Radical homogeneous ideals } I \subset B(\Sigma) \subset S\}.$$

Remark 10.7. As we explained in Remark 9.11, the symplectic toric manifold has deep relations with toric variety. The quotient construction, related to the GIT quotient theory (but we didn't explain in what sense), will correspond to the symplectic reduction (in fact, Kahler reduction) construction of symplectic toric manifold. The complement of the irrelevant ideal is homotopy equivalent to a subset of it, which is given by a level set of the moment map of a Hamiltonian action on the complement of the irrelevant ideal. The quotient of both construction are the same follows from the Kempf-Ness theorem.

Example 10.8. For $\mathbb{P}^n$, its quotient construction is given by the quotient $\mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times$. Here, we consider the circle action that restricts from the $\mathbb{C}^\times$ action, which is Hamiltonian with a moment map given by

$$\mu(x_0, x_1, \cdots, x_n) = x_0^2 + x_1^2 + \cdots + x_n^2.$$

Then $\mu^{-1}(1) = S^{2n+1}$ and the inclusion $\mu^{-1}(1) \subset \mathbb{C}^{n+1} \setminus \{0\}$ induces a diffeomorphism $\mathbb{P}^n = \mu^{-1}(1)/S^1 \simeq \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times$.

11. Fano toric variety and Calabi-Yau hypersurfaces

This section, we discuss Batyrev's construction for CY hypersurfaces in Fano toric varieties. We refer to [Bat93], but mostly follows notation in this notes. All lattice polytopes in this section are assumed to be full dimensional ($= n$).

We start from characterization of toric Fano varieties. For a lattice polytope Δ, we say it is ***reflexive*** if its facet presentation has the following form

$$\Delta = \{m \in M_{\mathbb{R}} : \langle m, u_F \rangle \geq -1, \, F \text{ facets of } \Delta\}.$$

Equivalently, it means that 0 is the only interior lattice point of Δ.

Definition 11.1. We say a normal variety X is Gorenstein if the anti-canonical divisor $-K_X$ is Cartier, it is Gorenstein Fano if the anti-canonical divisor $-K_X$ is Cartier and ample.

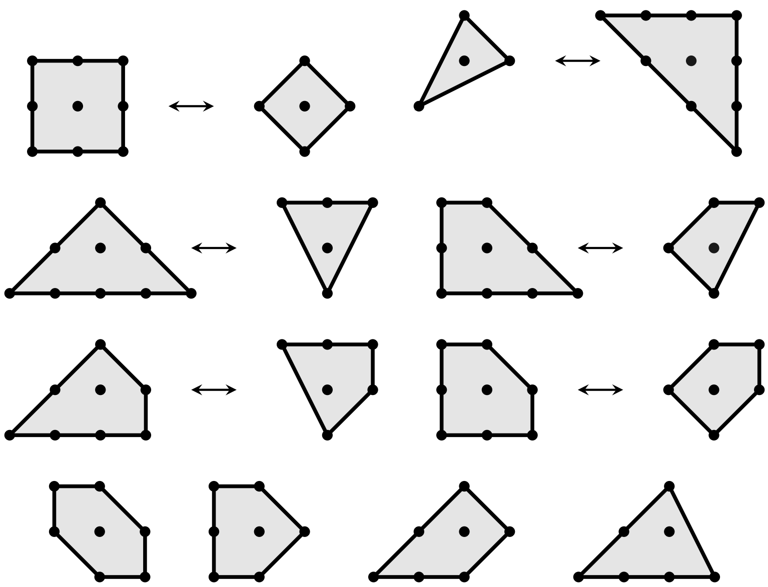
Proposition 11.2. *For a lattice polytope Δ, we have*

$$X_\Delta \text{ is Gorenstein Fano} \quad \Longleftrightarrow \quad \Delta \text{ is reflexive.}$$

Proof. If X_Δ is Gorenstein Fano, then by definition of Fano, we have that $-K_X$ is Cartier and ample. We use $-K_X$ as an ample divisor to define projective embeddings with associated polytope P_{-K_X}. But we have that $P_{-K_X} = \Delta$ upto $\mathrm{GL}_n(\mathbb{Z})$, and then Δ is reflexive by the formula of P_{-K_X} with the help of Corollary 8.7.

Conversely, if Δ is reflexive, then $\Delta = P_{-K_X}$ by Corollary 8.7 and $-K_X = D_\Delta$ is Cartier by Proposition 9.6 and ample by Theorem 9.9. $\square$

Example 11.3. In dimension 2, there are 16 equivalent classes of reflexive lattice polygons upto $\mathrm{GL}_2(\mathbb{Z})$. We see that most of them can be constructed from the $\mathbb{P}^2$ polygon by removing corners. By Example 9.5-(4), this means that most of toric Fano surfaces can be constructed by successive blow-ups of $\mathbb{P}^2$. However, they do not cover all Fano surfaces.



All 16 reflexive 2-dimensional polytopes [Hof18, Figure 1.5] and their dual pairing.

Now, we can study CY hypersurfaces. We first introduce definitions: We say a smooth algebraic variety is Calabi-Yau if its its canonical bundle is trivial[10]. For a resolution of singularities $\phi : X' \to X$, we say it is *crepant* if $\phi^* K_X = K_{X'}$ as divisor classes [11].

[10] In literature, definitions of Calabi-Yau are not uniform. Here, we consider the loosest requirement (but the most common), which allows non-compact and/or non simply connected examples.

[11] To make sense the notation here, we need to assume X is Gorenstein (or $\mathbb{Q}$-Gorenstein) to make sure K_X is Cartier. Then we define the pull-back via pull-back of like bundle.

Now, for a reflexive lattice polytope Δ, we assume X_Δ has a toric crepant resolution $\phi : X_\Sigma \to X_\Delta$. Then $\phi^* K_{X_\Delta} = K_{X_\Sigma}$ is basepoint free. We pick a generic section s of $\mathcal{O}_{X_\Sigma}(-K_\sigma)$ such that the zero locus V_Δ of s is smooth and irreducible; V_Δ is of dimension $n-1$.

Proposition 11.4. *We have that V_Δ is Calabi-Yau and $h^{i,0}(V_\Delta) = 0$ for $0 < i < \dim V_\Delta$.*

Proof. Calabi-Yau property comes from adjunction formula since X_Σ is Fano.

We will not use the cohomological vanishing here, we refer to [CLS11, Proposition 11.2.10] for readers. □

Remark 11.5. In fact, we should think V_Δ as a family of CY hypersurfaces since we pick general hypersurfaces. We still use V_Δ as the family of hypersurfaces and this is the family $\mathcal{F}(\Delta)$ in [Bat93].

Remark 11.6. The essential problem here is that we do not know if crepant resolution exists in general (generally known for $n \leq 3$ and unknown for $n \geq 4$). Batyrev do something more sophisticated (which produces singular Calabi-Yau in general) and it beyond the scope of this lecture. We just present this over simplified discussion here that sketches the idea of the story.

Finally, we want to roughly discuss the Mirror phenomenon of Batyrev CYs. It based on the following simple observation: For a reflexive lattice polytope, we define its dual as

$$\Delta^\vee = \mathrm{Conv}(u_F : F \text{ facets of } \Delta).$$

Lemma 11.7. *For a full dimensional reflexive lattice polytope Δ, its dual $\Delta^\vee$ is also a full dimensional reflexive lattice polytope. Moreover, we have $\Delta = (\Delta^\vee)^\vee$.*

Therefore, for a given full dimensional reflexive lattice polytope Δ, we have a pair of toric Fano varieties $(X_\Delta, X_{\Delta^\vee})$. If both of them have crepant resolutions, then our receipt produces a pair of CY hypersurfaces $(V_\Delta, V_{\Delta^\vee})$. They are expected to mirror with each other in certain sense.

Remark 11.8. We know that u_F are ray generators of the normal fan Σ_Δ. Then $\Delta^\vee$ is actually the convex hull of the ray generators of Σ_Δ. Therefore, in some (particularly physical) literature, people would say $\Delta^\vee$ produce X_Δ. In some sense, if you thought $\Delta^\vee$ as a fan, then nothing wrong; but in our convention, it suppose to produce $X_{\Delta^\vee}$. This would make a lot of confusions!!!

Example 11.9. In this example, we cook up the famous Greene–Plesser quintic 3-CY mirror pair. (Here, we refer to [CLS11, Example 5.4.10] rather than Batyrev's paper.)

We start from Δ considered as a lattice polytope in $M = \mathbb{Z}^4$, we denote Δ_4 the unit simplex in $\mathbb{R}^4$, then we set $\Delta = 5\Delta_4 - (1,1,1,1) \subset M_\mathbb{R}$. Its vertices are

$$v_0 = (-1,-1,-1,-1), v_1 = (4,-1,-1,-1), \cdots, v_4 = (-1,-1,-1,4) \in M.$$

Clear we have $\Sigma_\Delta = \Sigma_{\mathbb{P}^4}$ (we see its ray generators are u_i described below), $X_\Delta = \mathbb{P}^4$, $D_\Delta = -K_{\mathbb{P}^4}$. Its anti-canonical sections gives generic quintic V_Δ, which is Calabi-Yau. In fact, this family has 101-parameters (in fact $126-24-1$, try to understant what are those numbers), and we often just write the Fermat type

$$x_0^5 + \cdots + x_4^5 = 0$$

as a representative (and say it is mirror to the 1-parameter family we present below).

The dual $\Delta^\vee \subset N_\mathbb{R}$ itself is spanned by

$$u_0 = (-1,-1,-1,-1), u_1 = (1,0,0,0), \cdots, u_4 = (0,0,0,1) \in N.$$

Ray generators of its normal fan $\Sigma_{\Delta^\vee}$ (in M) are vertices of $\Delta \subset M_\mathbb{R}$: v_i we described as above.

We consider the sublattice M_1 generated by u_i in M. We claim that

$$M/M_1 \simeq \{(a_0,a_1,a_2,a_3,a_4) \in (\mathbb{Z}/5)^5 : \sum a_i = 0\}/\{(a,a,a,a,a) : a \in \mathbb{Z}/5\} \simeq (\mathbb{Z}/5)^3.$$

Because $M_1 \subset M$ is a sublattice, we have that $(M_1)_{\mathbb{R}} = M_{\mathbb{R}} = \mathbb{R}^4$. Then we can think $\Sigma_{\Delta^\vee}$ as a fan for both M_1 and M. However, with respect to M_1, the resulting toric variety is $X_{\Sigma_{\Delta^\vee}, M_1} = \mathbb{P}^4$; and with respect to M, the resulting toric variety is $X_{\Delta^\vee}$. Then it is proven that in Example 6.3-(3) that

$$X_{\Delta^\vee} \simeq X_{\Sigma_{\Delta^\vee}, M_1}/(M/M_1) = \mathbb{P}^4/(\mathbb{Z}/5)^3.$$

We can identify (by direct computation) this $(\mathbb{Z}/5)^3$ in the quotient with

$$\{(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4) : \prod \mu_i = 1\}/\{(\mu, \mu, \mu, \mu, \mu)\}$$

for the fifth root of unit μ_i, μ, and it acts on $\mathbb{P}^4$ diagonally.

Remark 11.10. Using the quotient construction or projective embedding, we can also describe $X_{\Delta^\vee} = \mathbb{P}^4/(\mathbb{Z}/5)^3$ as the hypersurface $Y_1Y_2Y_3Y_4Y_5 = Y_0^5$ in $\mathbb{P}^5$ ($\ni [Y_0, Y_1, Y_2, Y_3, Y_4, Y_5]$). We left the detail to readers.

In fact, the above procedure for introducing the sublattice M_1 actually gives a toric crepant resolution: it is direct to see that $\mathbb{P}^4 \to \mathbb{P}^4/(\mathbb{Z}/5)^3$ pull-back canonical divisor to canonical divisor (combinatorially, this is because when considering $\Sigma_{\Delta^\vee}$ as fans in both M_1 and M, we only reparameterize the lattices, but no extra ray generators are added). Moreover, the resolution commute with taking anti-canonical hypersurfaces.

So, the resulting CY hypersurfaces $V_{\Delta^\vee}$ in $\mathbb{P}^4$ is exactly the quintic that can descent to $\mathbb{P}^4/(\mathbb{Z}/5)^3$. To see what quintic can descent to $\mathbb{P}^4/(\mathbb{Z}/5)^3$, we look at the anti-canonical divisor of $\mathbb{P}^4/(\mathbb{Z}/5)^3$ that corresponds to the polytope $\Delta^\vee$ and correspondent generic anti-canonical hypersurfaces in $\mathbb{P}^4/(\mathbb{Z}/5)^3$. They should have Newton polytope $\Delta^\vee$, and can be given by the following Laurent polynomial

$$\frac{c_0}{t_1t_2t_3t_4} + c_1t_1 + c_2t_2 + c_3t_3 + c_4t_4 + c_5 = 0;$$

We take homogenization respect to $-K_{\mathbb{P}^4/(\mathbb{Z}/5)^3} = D_{\Delta^\vee}$ (defined in Section 10): For $n \in N = \mathbb{Z}^4$, the character $\chi^n(t) = t^n$ corresponds to a monomial of y_i with exponent vector

$$\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} n + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} n + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We plug $n = u_i$ to get the c_i term. Then the resulting homogeneous equation in variable y_i is

$$c_0y_0^5 + c_1y_1^5 + c_2y_2^5 + c_3y_3^5 + c_4y_4^5 + c_5y_0y_1y_2y_3y_4 = 0,$$

which is clear to see that it descent to $\mathbb{P}^4/(\mathbb{Z}/5)^3$ by the $(\mathbb{Z}/5)^3$ action we describe above. Then we claim that the resolve CY hypersurfaces $V_{\Delta^\vee}$ in the resolved toric Fano $\mathbb{P}^4$ (you should think it as $X_{\Sigma_{\Delta^\vee}, M_1} = \mathbb{P}^4$ rather then $X_{\Sigma_\Delta, N} = \mathbb{P}^4$) is given by this quintic.

For generality, we could require $c_0, c_1, c_2, c_3, c_4 \neq 0$. Notice that the $\mathbb{T} = \mathbb{T}_4$ acts on the resolution $\mathbb{P}^4 \to \mathbb{P}^4/(\mathbb{Z}/5)^3$ transitively, we can change coordinates. Then we assume that $c_0 = c_1 = c_2 = c_3 = c_4 = 1$ (one because the equation is homogeneous, four because of the $\mathbb{T}$-action). Then the resulting hypersurfaces $V_{\Delta^\vee}$ could be written as (we set $c_5 = \psi$)

$$\{y_0^5 + y_1^5 + y_2^5 + y_3^5 + y_4^5 + \psi y_0y_1y_2y_3y_4 = 0\} \subset \mathbb{P}^4 = X_{\Sigma_{\Delta^\vee}, M_1}.$$

This gives the familiar equation in literatures, and the mirror CY3 $V_{\Delta^\vee}$ (or if you somehow allow singular Calabi-Yau, it is also reasonable to call the hypersuraces in $\mathbb{P}^4/(\mathbb{Z}/5)^3$ as the mirror). $\square$

12. Coherent-Constructible correspondence

We refer to [She22], and we follow directly the (very short) article.

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